GEOMETRY.

GEOMETRY.

PRINTED BY
SPOTTISWOODE AND CO., NEW-STREET SQUARE
LONDON

FIRST STEPS

IN

GEOMETRY:

A SERIES OF HINTS FOR THE SOLUTION OF

GEOMETRICAL PROBLEMS

WITH

NOTES ON EUCLID, USEFUL WORKING PROPOSITIONS
AND MANY EXAMPLES.

BY

RICHARD A. PROCTOR,

AUTHOR OF 'CHANCE AND LUCK,' 'EASY LESSONS IN THE DIFFERENTIAL
CALCULUS,' 'THE GEOMETRY OF CYCLOIDS,' AND THE ARTICLES
ON ASTRONOMY IN THE 'ENCYCLOPÆDIA BRITANNICA'
AND THE 'AMERICAN CYCLOPÆDIA.'

'The work about the square on't.'—*Shakespeare.*

LONDON:
LONGMANS, GREEN, AND CO.
AND NEW YORK: 15 EAST 16th STREET.
1887.

PREFACE.

THE object I have had in view in preparing this little work (which appeared first in the pages of KNOWLEDGE) has been to remove for young students in geometry the difficulties which I remember encountering when a beginner myself. Teachers and books explained then, as now, how certain problems are to be solved, but they did not show how the student was to seek for solutions for himself. They strove to impart readiness in following demonstrations rather than facility in obtaining solutions. My method of showing here why such and such paths should be tried, even though some may have to be given up, in searching for the solution of problems, will, I believe, do more to teach the young student how to work out solutions for himself than any number of solutions given him for reading.

The notes to the first two books of Euclid and added propositions—a knowledge of which is absolutely essential for success in solving problems—are subsidiary to the purpose of this little treatise. The similar study of later books may be commended to students more advanced than those for whom I have written here.

RICHARD A. PROCTOR.

ST. JOSEPH, MO.: *May* 1887.

CONTENTS.

SECTION I.

GEOMETRICAL PROBLEMS.

SECTION II.

NOTES ON EUCLID.

SECTION III.

RIDERS AND PROBLEMS ON THE FIRST TWO BOOKS.

SECTION I.

GEOMETRICAL PROBLEMS.

INTRODUCTION.

THE object of these hints to the solution of geometrical problems is to show the student how he should deal with deductions which are proposed to him in examination. There are many books in which sets of problems are given, with several fully solved, and hints supplied for the solution of others; but these are often of little use to the student. The average mathematical student requires to learn—not how to solve this or that problem, nor what construction will help him in any particular case: but what are the general methods which he must apply to problems in order to obtain solutions for himself. The mathematical teacher who simply solves the problems brought to him by his pupils does little to show how such problems are to be treated. He should exhibit to his pupils the train of thought which leads him to apply such and such processes to the solution of a problem. And more than this: a good tutor will show his pupils where they might be led astray by imperfect methods; he will try the effects of steps which he himself knows to be bad, and thus show his

pupils what methods to avoid as well as what methods to apply. One problem thus dealt with is worth a dozen which are merely solved; and I believe the student who will carefully go through the examples which I shall take to pieces (so to speak) in the following series, will learn more than he would from seeing any number of problems merely solved.

I. Geometrical Deductions.

Geometrical deductions are problems which are intended to be solved by the application of recognised geometrical methods and propositions. They are divided into several classes.

A geometrical deduction is termed a *rider* when it is given as an exercise on a particular proposition. It generally happens that the difficulty of a deduction is greatly diminished when it is given in this way, for we know in what direction to seek for a solution. When a deduction is presented as a rider, it is, of course, expected that the proposition to which the deduction is appended shall be made use of in the solution. It will occasionally happen, with carelessly-constructed riders, that a simpler solution, not involving this proposition, is available; but generally there can be no difficulty in so arranging the proof as to introduce the proposition on which the deduction is supposed to be founded.

A deduction may be given as an exercise on a particular book of Euclid, or on a given set of propositions. In such a case, it is, of course, expected

that no later books or propositions (as the case may be) shall be made use of.

Or, a deduction may be given as an exercise on Euclid, generally—in which case it is expected that no *methods* which are not used by Euclid shall be applied to the solution of the problem ; and, further, that no proposition not contained in Euclid, or not readily deducible from Euclid's propositions, shall be made use of.

Lastly, there are deductions of a more advanced character, and propositions which present themselves in the solution of problems in other subjects, such as trigonometry, optics, mechanics, and so on. In treating deductions of this sort, it is allowable to make use of several well-known geometrical problems not established by Euclid, nor obviously deducible (that is, deducible as *corollaries*) from his propositions. Hence these properties may themselves be presented as exercises on Euclid—and in fact most of them will be found in collections of deductions. It seems better, however, to direct the student's attention specially to propositions of this sort, since their importance is apt to be lost sight of when they are included in a long list of deductions. It is possible that I may on some future occasion attempt to gather together all those propositions which may fairly be looked on as subsidiary. Some of them are very simple, others less so ; but the student should have all of them at his fingers' ends, since they are of continual service in geometrical processes.

II. CONSTRUCTION.

The first step in the solution of a geometrical problem is the construction of a figure which shall afford a clear conception of what we have to do or prove. There are some who insist that no one deserves to be called a geometrician who makes use of well-drawn figures. To solve a difficult problem when the illustrative figure is unlettered, or when ovals are drawn for circles, waved lines for straight ones, and so on, may be all very well for the advanced mathematician. Indeed, a good geometrician should be able to take up a list of problems and solve the major part without pen or paper. But it seems to me a great mistake to insist that the learner should increase the difficulties he naturally has to encounter by making difficulties for himself. And independently of this consideration, there is nothing better calculated to lead the student to observe new properties —or properties new to him—than the construction of a well-drawn figure. He is led to notice relations which would otherwise escape him. Thence he learns to seek for the proof of such relations, to satisfy himself that they are real—not apparent. And it is this habit of being always on the watch for new properties which serves as the most efficient aid in the solution of geometrical problems, and which, also, so far as mathematical progress is concerned, is the most valuable fruit of geometrical studies.

The beginner should even use mathematical

instruments, and should spare no pains in the exact construction of his figures. But after awhile, all that will be necessary is that the figures should be drawn, free-hand, so as to represent as closely as possible the relations described in the proposition to be investigated. Simple as this seems to be, there are some points which deserve to be attended to. A few illustrations will serve better than formal rules :—

Suppose a problem spoke of a trisected line: the student would probably draw a line, as A B (Fig. 1), and then divide it as nearly as possible into three equal parts, in C and D. This is not the best plan: he should draw a line, as A D, bisect it as nearly as possible in C, and then produce it to B, so that D B

FIG. 1.

may be as nearly equal to D C as possible. He will thus have a line much more exactly trisected than by the former method, since everyone can bisect a line, or produce it till the part produced is equal to the adjacent part, whereas many fail in the attempt to trisect a line. Similar remarks apply to the division of a line into five, seven, or nine equal parts.

Suppose we had to solve such a problem as the following:—*From a given point outside the acute angle contained by two given straight lines, to draw a straight line so that the part intercepted between the two given straight lines may be equal to the part between the given point and the nearest line.*

Here the natural process, in constructing the figure, would be to draw the lines A B and B C (Fig. 2), and taking P as the given point, to draw P D E, so that P D and D E might be as nearly equal as possible. The proper way, however, is to draw a straight line, P E, bisect it in D, and through the points D E to draw the lines A D B, C E B, meeting in B.

Again, suppose a problem spoke of a circle touching a given line in a given point, and passing

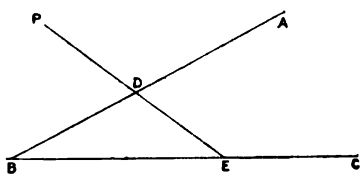


Fig. 2.

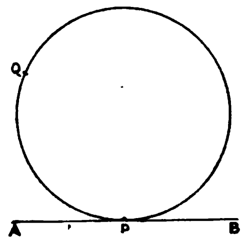


Fig. 3.

through another given point. Then we should not draw a straight line, and taking the points P and Q (Fig. 3), attempt to draw a circle through Q to touch A B in P. We should first draw the circle, then draw a tangent, A P B, and take a convenient point, Q, upon the circumference of the circle.

In like manner if, in a deduction, mention is made of a circle inscribed within, or circumscribed without, a triangle, we shall obtain a far more satisfactory figure by drawing the circle first, and then

forming a triangle round it or within it, respectively, than by drawing the triangle first.

These instances suffice to exhibit the necessity of considering the order of the constructions needed in our figure. There are some considerations to be attended to, also, respecting the *shapes* to be given to different figures, that an examination of the properties they are meant to illustrate may be made as easy to us as possible.

It is very important that the different parts of a figure should not exhibit apparent relations not really

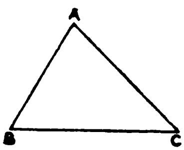


FIG. 4.

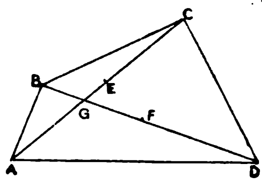


FIG. 5.

involved in the problem illustrated. Lines should not seem to be equal, or to be at right angles to each other, when they are not necessarily so. Triangles should not seem to be isosceles or right-angled when the problem does not involve such relations. It is well to notice that, in general, the most convenient form of triangle for illustrating general properties is that shown in Fig. 4; here the angle A is one of about 75°, the angle B one of about 60°, and the angle C one of about 45°. When a quadrilateral figure is not necessarily either a parallelogram or a trapezium, it is well to construct it of such a

figure as A B C D (Fig. 5), in which the four sides are unequal, neither pair of opposite sides parallel, and the diagonals A C, B D do not make equal angles with any side. It will be noticed, also, that neither diagonal bisects the other. If we had a problem in which the bisections E and F of the diagonals were concerned, all that would be necessary, in order that neither diagonal might bisect the other, would be to draw the diagonals A C and B D *first*, so that their point of intersection, G, should be well removed from the bisections E and F; then join A B, B C, C D, and D A.

It is sometimes convenient to draw a part of the figure in darker lines than the rest. We may distinguish in this way, for instance, between the lines or circles belonging to the enunciation and those belonging to the construction. When we are in doubt as to the necessity of any construction, it may be lightly dotted in. In very complex figures, dark, light, broken, and dotted lines may be conveniently employed together.

Always letter every point of the figure which may have to be referred to as you proceed. It is often as well, when a result has been established which seems to promise to be useful towards the solution of a problem, to re-draw the figure, omitting all lines except those which have served to guide you to this result. But, except in such instances, or where the figure seems obviously unsuited to your requirements, or has become overcrowded with constructions, it is

well to keep to the same figure as long as possible. The habit of repeatedly re-drawing figures interferes with the concentration of the attention and the steady progress from result to result, which alone avail toward the solution of difficult problems.

III. Analysis and Synthesis.

There are two general modes of treatment applicable to problems, termed, conventionally, the *synthetical* and the *analytical*, or *synthesis* and *analysis*. In the former, we study what is given and work up to what is sought; in the latter, we examine what is sought and work back to what is given. I am not concerned here with the correct applicability of the names 'synthesis' and 'analysis' to these processes, and shall therefore content myself with discussing the processes themselves under the names usually given to them.

It is a mistake to suppose that, as some have asserted, *analysis* is the method always employed—consciously or unconsciously—in the solution of problems. Of course, we are compelled to consider what it is we have to do or prove, and thus far the analytical method cannot but enter into our processes. But in the solution of a problem, we may proceed, as may be most convenient, by either the synthetical or the analytical process, or—which in complex problems is far more commonly the case—by an alternation of both methods. As an illustration of my

meaning, I may compare geometrical problems to those examples in algebra, trigonometry, &c., in which we have to establish the identity of two expressions. In such cases we may either take one expression, and try to work it into the same form as the other, or *vice versâ*, we may select the latter to work upon, or—which is the surer process—we may work both down to a common form.

However, it will be better to select a few examples of geometrical problems, and to exhibit the application of different processes to them, than to discuss general rules. I begin with very simple examples.

Suppose we have to deal with the following deduction:—

Ex. 1.—*The line A B (Fig. 6) is bisected in C, and C D is drawn at right angles to A B. From any*

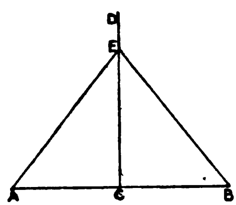


FIG. 6.

point E in C D lines are drawn to A and B. Show that E A is equal to E B.

Having constructed a figure in accordance with these data, we go over the data thus: we have A C equal to C B, and C E at right angles to A B. We

remember, also, that we have to prove that A E is equal to E B. Now, we cannot fail to see that the data involve the equality of the triangles A C E, C E B, by Euc. I., 4, and therefore that A E *is* equal to E B. This solution is synthetical, notwithstanding the prior reference to the relation which has to be established. For we proceed from the data—A C, C E, equal to B E, C E, and the included angles equal—to the equality of the triangles A C E, B C E in all respects, and thence to the equality of A E, E B. In the analytical solution we should argue thus:—We have to show that A E is equal to E B. Now, *if* A E is equal to E B, then since A C, C E are respectively equal to B C, C E, the angles A C E and B C E will be equal (Euc. I., 8); but these angles *are* equal, being right angles; hence we are led to reverse the steps as a *probable* method of solving our problem; and, on trial, we find that the proof of the equality of A E, E B, is complete by this method. We shall presently see that the mere fact of obtaining by the analytical method a result corresponding to certain data of a proposition is no certain test that the problem is correct; and I will at once show that it is no certain proof that the reversal of the process will give at once a satisfactory solution of a problem.

Suppose that we have given to us A C equal to C B, and the angle C A E equal to the angle C B E, and that we have to show from these data that C E is at right angles to A B. We proceed analytically thus: *If* C E is at right angles to A B, then A C,

C E, being equal, respectively, to B C, C E, the triangles A C E, B C E are equal in all respects; therefore the angle C A E will be equal to the angle C B E. Now, these angles *are* equal; therefore we might expect the reversal of the process to lead at once to the solution of our problem. This, however, is not the case—we have A C, C E equal to B C, C E, and the angles C A E, C B E, *opposite* to the common side, C E, equal to each other; but there is no proposition in Euclid which enables us to assert from these data that the triangles C A E and C B E are equal in all respects.

Of course, there is no difficulty in the above problem. The equality of the angles C A E and C B E give us immediately A E equal to E B (Euc. I., 6), and thence the equality of the triangles, A C E, B C E, follows at once. But it is well to notice that analysis may lead to a result involved in our data which yet does not involve the immediate solution of our problem.

Let us take next a less obvious proposition:—

Ex. 2.—*In the figure to Euc. I.*, 5 (*Fig.* 7), *if B G, C F intersect in H, show that A H bisects the angle B A C.*

Let us go over our data:—We have A B equal to A C, the angle A B C equal to the angle B C A, and also (see the proof of Euc. I., 5), the angle A B G equal to the angle A C F, and the angle G B C equal to the angle B C F. There are other relations which seem unlikely to aid us, so we content ourselves with

these. Remembering that we have to prove the equality of the angles B A H and C A H, we are at once led to notice that our data point to the equality of the triangles H B A and H C A. For we have the angle A B H equal to the angle A C H, and also A B, A H equal to C A, A H, respectively. But these relations are not sufficient. Seeing, however, the probability that the solution of our problem lies in this particular direction, we search for some new equality in the elements of the triangles A B H, C A H.

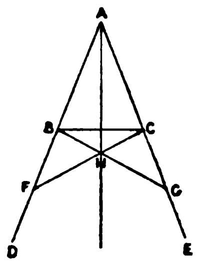


FIG. 7.

Can we, for instance, show that the angle A H B is equal to the angle A H C? This seems no easier than to establish the equality of the angles H A B, H A C. Can we, then, prove the equality of the sides H B, H C? This would involve the equality of the angles H B C, H C B (Euc. I., 6); and this is one of our data. Hence we see our way at once to the solution of the problem, which runs thus :—

Since the angle H B C is equal to the angle H C B, H B is equal to H C. Hence in the triangles

B A H, C A H, we have B A, A H equal to C A, A H, each to each, and the base B H equal to the base C H. Therefore the angle B A H is equal to the angle C A H. (Euc. I., 8.)

It will be noticed that the reasoning from which this solution is obtained is partly synthetical and partly analytical. We apply our data to obtain a result which very nearly gives us what we want; then we inquire analytically how the missing link is to be supplied; and finally, having seen our way to the solution, we run over such portions of our reasoning as are required for the complete proof of the proposition. The mental process is, of course, considerably longer than the solution which results from it—the mind runs rapidly over the elements given and required, selecting and rejecting this or that relation until the path to the complete solution has been traced out. I have only followed one such process of reasoning—that which seems to me most natural. Others might readily be conceived. Thus the equality of the lines A F, A G, and the angles A F C, A G B (see the proof of Euc. I., 5) might occur as the most obvious data for selection. It would, then, be seen that before we can establish the equality of the triangles, F A H, G A H, we must prove that F H is equal to H G; but we know that F C is equal to B G; therefore, we must prove that the remainder, H C, is equal to the remainder, H B. This requires the equality of the angles, H B C, H C B. We know these angles to be equal; there-

fore, H C is equal to H B, and thence F H to H G; and since A F, A H are equal to G A, A H, the angle F A H is equal to the angle G A H. It is probable, however, that the geometrician, being led in *this* way to the equality of H B and H C, would not retrace the steps he had followed, but would immediately notice the shorter proof depending on the equality of B A, A H to C A, A H, respectively.

Thus we gather an important rule. Having tracked out, analytically or synthetically, a complete proof of a proposition, it is well before writing down the solution to notice whether the relations which have presented themselves in the process of reasoning suggest a shorter proof, or whether any of the steps of the reasoning may be omitted, or so varied as to be reduced in number. The value of a proof is, of course, much enhanced by brevity and conciseness.

IV. THEOREMS.

We have hitherto taken theorems involving *exact* results as our illustrative examples, and we have seen that to such theorems, analytical or synthetical methods, or combinations of both, are applicable with equal advantage. We shall presently discuss other propositions of this sort, and of greater complexity. But we must now notice the fact that in certain propositions we have no choice as to the method of solution. This is almost always the case with theorems involving *general* results, and with *problems* properly so called—that is, with propositions in which some-

thing is required to be *done*. Propositions of the former class require the synthetical, propositions of the latter class the analytical method. But of course neither of these rules holds, necessarily, in problems of great simplicity, in which only one or two steps separate the *data* from what is sought.

We begin with an instance of this sort—viz. a very simple theorem in which the relation to be established is general. Suppose we have to prove the following proposition:—

Ex. 3.—*Let A B C (Fig. 8) be an isosceles triangle, A B being equal to A C: produce A B to D, and join D C. Then shall D C be greater than B C.*

There is only one proposition in Euclid which deals with the inequality of two sides of a triangle —viz. Prop. 19, Bk. I. It naturally occurs to us, therefore, to apply this proposition. We have to show that D C is greater than B C, and we know from Euc. I., 19, that if D C *is* greater than B C, then the angle D B C is greater than the angle B D C. Now, our figure shows us D B C as an obtuse angle, and a moment's consideration shows that D B C is necessarily obtuse. For this requires that A B C should be necessarily acute. But the angle A B C is equal to the angle A C B (Euc. I., 5), and two angles of a triangle being less than two right angles (Euc. I., 17), each of these angles must be less than one right angle. Therefore A B C is acute, and its supplement, D B C, is obtuse. B D C is therefore acute, and D C greater than B C.

We will next try a proposition slightly more difficult.

Ex. 4.—*Let P (Fig. 9) be a point which does not lie on either diagonal of the quadrilateral A B C D; then shall the sum of the four lines A P, B P, C P, and D P be greater than the sum of the diagonals A C, B D.*

Here it would serve us nothing to begin analytically by *supposing* A P, B P, C P, and D P to be

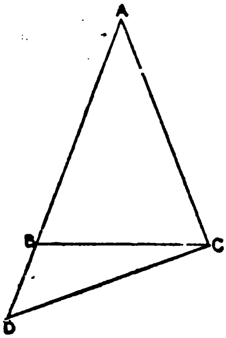


FIG. 8.

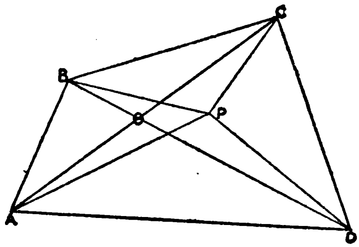


FIG. 9.

together less than A C and B D together. It would be impossible to deduce anything from a general relation of that sort. We must therefore proceed synthetically.

Let O be the intersection of the lines A C, B D.

Then we might first be struck by the fact that B P and A P are together greater than A O and O B together (Euc. I., 21). But then we notice that, on the other hand, C P and P D are together less than

C O, O D together. So that unless we can show the former *excess* to be greater than the latter *defect*, we have proved nothing to our purpose. This method does not seem promising. Nor does it seem likely to be useful to take B P, P C together and then A P, P D together, comparing these pairs, respectively, with B O, O C together and A O, O D together.

Let us try taking alternate lines together, namely, B P, P D, and A P, P C. We at once see that B P, P D are together greater than the diagonal B D; and that A P, P C are together greater than A C. Hence, P A, P B, P C, and P D are together greater than A C and B D together.

We will now try a problem of less simplicity though by no means difficult.

Ex. 5.—*The triangles B A C, B D C* (*Fig.* 10) *are on the same base, B C, and between the same parallels, A D and B C; also, B A C is isosceles, the side B A being equal to the side A C. Show that the perimeter of the triangle B A C is less than the perimeter of the triangle B D C.*

First of all we notice that B C being common to both triangles, we need only prove that B A, A C are together less than B D, D C together.

After a little examination it becomes clear that the sides B A, A C are not easily comparable with the sides B D, D C, *as they stand.* It is an obvious resource to produce B A to A F, making A F equal to A C, so that B F is equal to the sum of the lines B A, A C; and in like manner to produce B D to G, making

D G equal to D C, so that B G is equal to the sum of the lines B D, D C. We have, then, to show that B F is less than B G. If B F *is* less than B G, then joining F G, the angle B G F is less than the angle B F G (Euc. I., 19). But there seems no obvious method of proving this relation.

Let us consider our construction. We have A F equal to A C. But A C is equal to A B. Thus B A,

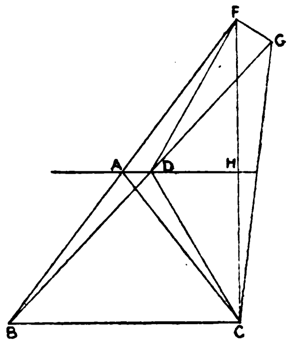


FIG. 10.

A C, and A F are all equal. Hence a circle described with centre A, through B, would pass also through C and F. Since B F would be the diameter of this circle, B C F is a right angle. (Euc. III., 31.) We therefore join C F, and note that it is perpendicular to B C.[1]

[1] Here is an instance of the advantage of carefully constructed figures. The relation arrived at by a tolerably obvious line of reasoning *might* be overlooked for awhile. But if the figure has been constructed carefully, in accordance with the data, the uprightness of F C could not escape notice, and a moment's inquiry

We notice, also, that C A F is an isosceles triangle. Let us see, then, about B G. We have D G equal to D C. But D C is less than D B, since the angle D B C is less than the angle D C B. (Euc. I., 18.) This does not seem likely to help us. If we join G C, C D G is, like C A F, an isosceles triangle. But this, again, does not appear, on examination, to be a profitable relation.

Let us see, however, whether we are making use of all our data:—We have been forgetting that A D is parallel to B C. Without making use of this relation we cannot hope to solve our problem.

We have shown that C F is at right angles to B C, and therefore, of course, C F is at right angles to A D. It will be as well, therefore, to produce A D to meet F C in H, and to note that A H is at right angles to F C. But C A F is an isosceles triangle, and it is a well-known property that the line drawn from the vertex of an isosceles triangle, at right angles to the base, bisects the base.[1] Thus C H is equal to H F. Will this property help us? Let us consider: C H is equal to H F, and H D A is at right angles to C F. Clearly, then, if we join D F, we have D F equal to D C, for the triangles D H C and D H F will be equal in all respects. (Euc. I., 4.)

would show that it is not accidental and suffice to exhibit its cause.

[1] This problem is not explicitly stated in Euclid. It is contained implicitly in Bk. I., Props. 10–12. It should be included amongst the additional problems a knowledge of which is necessary to those who wish to work successfully at deductions.

D F being equal to D C, we may be led to proceed in one of two ways :—

First, we might notice that our construction made D G equal to D C; so that D F is equal to D G, therefore the angle D F G equal to the angle D G F (Euc. I., 5); therefore the angle B F G greater than the angle B G F; and B G greater than B F (Euc. I., 19); that is, B D, D C, together greater than B A, A C together.

Or, we might notice that since D F is equal to D C, B D and D F are together equal to B D and D C together; but B D and D F are together greater than B F (Euc. I., 20); therefore B D and C D are together greater than B A and A C together.

If we had followed the first of these courses, we should still scarcely fail to notice *afterwards* that the second is an available and a better solution.

We proceed, then, to run over such steps of the above work as are necessary to the proof of the proposition. In doing this we notice that a property of the third book has been made use of in proving that C F is at right angles to A H. We will assume that the proposition has been given as a deduction from the first book. Then, although the student might mentally have followed the course we have adopted, it would be well for him to modify the proof so as to avoid the use of Book III. This is easily done. The student sees at once that the proof involves the equality (in all respects) of the triangles A H F, A H C. He had the angle A F H equal to the angle

A C H, A F equal to A C, and A H common. This is not quite sufficient (so far as Euclid's treatment of triangles extends). But it is easy to supplement these data by establishing the equality of the angles F A H, H A C, these angles being respectively equal to the equal angles A B C, A C B (Euc. I., 29). Hence the triangles H A F, H A C are equal in all respects.

The construction and proof of the proposition we are dealing with run, therefore, thus:—

Produce B A to F, making A F equal to A C. Join D F, D C, and let A D, produced if necessary, meet F C in H. Then the angle F A H is equal to the interior angle A B C (Euc. I., 29). But A B C is equal to A C B (Euc. I., 5), and A C B to C A H (Euc. I., 29). Therefore the angle F A H is equal to the angle C A H. Also F A, A H are equal to C A, A H respectively. Therefore the triangles F A H, C A H are equal in all respects (Euc. I., 4). Hence F H is equal to H C, and the angles at H are right angles. Thus the triangles D H F and D H C are equal in all respects (Euc. I., 4). Therefore D F is equal to D C. But B D and D F are together greater than B F (Euc. I., 20), that is, than B A, A F together. Therefore B D and D C are together greater than B A and A C together; and the perimeter of B D C is greater than the perimeter of B A C.

V. Problems.

Let us next try a few problems—properly so termed—that is, propositions in which something is required to be done. In these, as we have said, the analytical method is nearly always to be preferred. We will begin with a simple example.

Ex. 6.— *On a given straight line describe an isosceles triangle, each of whose equal sides shall be double of the base.*

Let A B (Fig. 11) be the given straight line.

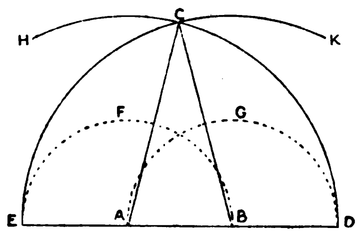


Fig. 11.

Suppose that what is required is done, and that on the base A B there has been described the triangle A C B, in which the sides A C and C B are equal to each other, and each double of the base A B; and let us consider what construction is suggested.

It seems hardly possible that the resemblance between this problem and Euc. I., 1, should escape the student's notice. He will inquire, then, whether the method of that problem cannot be applied to the present one. Instead of the circle with radius equal

to A B, we now require circles with radius equal to twice A B. It is clear, then, that if we produce A B to D, making B D equal to A B, and B A to E, making A E equal to A B (Euc. I., 3), then A D and B E will each be double of A B.[1] Therefore if with centre A and radius A D we describe a circle D C H, and with centre B and radius B E the circle E C K, then C, the intersection of these circles, is the vertex of the required triangle. For A C and B C are severally equal to A D and E B—that is, are double of the base, A B.

We will next try the following:—

Ex. 7.—*The point P (Fig. 12) is within the acute angle formed by the lines A B and A C. It is required to draw through P a straight line which shall cut off equal parts from A B and A C.*

Let D P E be the required line, so that A D is equal to A E.[2]

Then D A E is an isosceles triangle, and it is an obvious course to see whether any of the properties of isosceles triangles will help us to a solution of our problem. Now, the only property of isosceles triangles explicitly contained in Euclid is that of Book I.,

[1] We have seen this problem given with the proviso that no problem beyond Euc. I., 1, shall be made use of. In this case the student will see at once that if, with centres A and B, and distance A B, he describes the circles B F E, A G D, then E B and A D, the diameters of these equal circles, are severally double of A B.

[2] In constructing the figure, proceed thus:—Take A D equal to A E, and join D E; then *take P, a point dividing D E into unequal parts.*

Prop. 5. This gives us the angle A D E equal to the angle A E D—a property which avails us nothing.

But there are other properties of isosceles triangles, not expressly mentioned by Euclid, with which every geometrician ought to be acquainted. We will assume that the student is familiar with them—and indeed they are nearly self-evident. They are included in the statement that the perpendicular from the vertex on the base of an isosceles triangle bisects the base and also the vertical angle. Draw

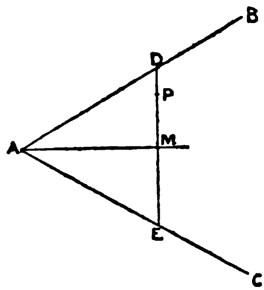


FIG. 12.

A M perpendicular to *the assumed line* D E; then the angle M A D is equal to the angle M A E, and also D M is equal to M E.

Now let us consider whether this construction affords us any hints.

First, we cannot see how to draw the line through A perpendicular to *the real line* D E, because it is this very line we seek to draw.

Secondly, we cannot, for a similar reason, see how to draw the line from A to the bisection of D E.

But, thirdly, we *can* draw the line A M bisecting the angle D A E.

And this clearly gives us the solution of our problem, since we can now draw D P E at right angles to A M. Thus the solution runs as follows:—

Draw A M bisecting the angle D A E, and through P draw D P M E at right angles to A M; then shall A D be equal to A E. For, in the triangles M A D, M A E, the angle M A D is equal to the angle M A E, the right angle A M D is equal to the right angle A M E, and A M is common to the two triangles; therefore the triangles are equal in all respects (Euc. I., 26), and A D is equal to A E.

The proof of the equality of the triangles M A D and M A E was not included in the prior examination of the problem, since it is involved in the assumed knowledge on the student's part of the fundamental properties of isosceles triangles, proved farther on. But of course it is well (in a case of such simplicity) to introduce the proof into the solution of the problem.

Let us next try the following problem:—

Ex. 8.—*The points P and Q (Fig. 13) are on the same side of the line A B. It is required to determine a point C in A B, such that the lines P C, Q C may make equal angles with A B.*

Let C be the required point, so that the angle P C A is equal to the angle Q C B.[1]

[1] Construct as follows: Draw A B, and from any point C in A B draw the *unequal* lines C P, C Q equally inclined to A B.

Let us try drawing a line, C D, at right angles to A B. Then the angle P C D is equal to the angle Q C D. On a consideration of this relation, however, it seems unlikely to help us. For it is not easier to gather anything from the equality of P C D and Q C D, than to make use of the equality of P C A and Q C B.

It seems an obvious resource, since the equality of the angles P C A and Q C B, as they stand, is not

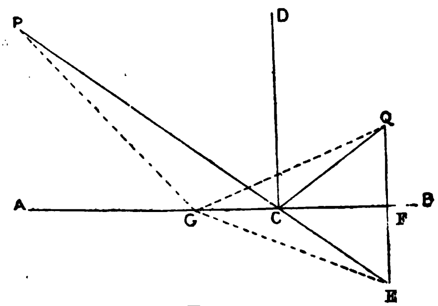


FIG. 13.

readily applicable to our purposes, to produce either P C or Q C, in order to see whether the vertical angle either of P C A or Q C B might be more serviceable to us. Produce P C to E. Then the angles Q C B and B C E are equal, or C B is the bisector of the angle Q C E. The only property connected with the bisector of an angle which seems likely to help us is this one, that the bisector of the vertical

Then there is no risk that accidental relations will appear as necessary ones.

angle of an isosceles triangle is perpendicular to and bisects the base. Now, we can make an isosceles triangle of which C shall be the vertex and C Q a side, for we have only to take C E equal to C Q, and to join Q E, cutting C B in F. Then, by the property just mentioned, Q E is at right angles to C F, and is bisected in F.

These relations obviously supply all we want. For, reversing our processes, we have only to draw Q F E perpendicular to A B, and to take F E equal to Q F; then drawing P E to cut A B in C, we are certain that C is the required point. In *all* such cases we should not be equally certain that the *proof* would be as simple as the analysis, since sometimes the reversal of a process involves properties not so readily seen as their converse theorems. In this case, however, it is obvious (or will at least appear so on a moment's inquiry) that the proof is simple.

For, join C Q (we are going now through the synthetic treatment of the problem, and therefore ignore the prior constructions), then, because Q F is equal to F E, and C F is common and at right angles to Q E, the triangles C F Q and C F E are equal in all respects. Therefore, the angle Q C F is equal to the angle E C F. But E C F is equal to the vertical angle P C A. Therefore the angle Q C F is equal to the angle P C A.

It is an excellent practice, when a problem has been solved, to notice results which flow from, or are in any way connected with, our treatment of the

problem. In Ex. 8 we notice that the line C Q (Fig. 13) is equal to the line C E, so that the sum of the lines P C, C Q is equal to the line P E. It might occur to us to inquire what is the sum of lines drawn from P and Q to any other point, as G, in A B. Join P G and Q G. The fact that C E is equal to C Q reminds us that if we join G E, G E will be equal to G Q. Thus P G and G Q are together equal to P G and G E together. But P G and G E are together greater than P E; that is, P G and G Q are together greater than P C and C Q together; or *P C Q is the shortest path from P to Q, subject to the condition that a point of the path shall lie on A B.*

VI. Problems on Maxima and Minima.

The result last obtained fitly introduces us to an important class of problems—viz. those in which we have to show that certain lines, areas, &c., are the greatest or least which can be constructed under certain assigned conditions. There are few problems of this sort in Euclid. In fact, the seventh and eighth propositions of the third book are the only theorems in Euclid expressly dealing with geometrical maxima and minima. But many interesting deductions involve such relations as we are speaking of, and it is well for the student to know how to deal with them.

It will be noticed that some of the problems already dealt with may be presented as examples of

geometrical maxima and minima. For instance, Ex. 4 may be presented in the following form :—

Ex. 9.—*From a point within a quadrilateral, lines are drawn to the angles of the quadrilateral: show that the sum of these lines will be a minimum when the point is at the intersection of the diagonals.*

Presented in this form the problem would be solved precisely as Ex. 4. But suppose it had been given in the following form :—

Determine a point within a quadrilateral such that the sum of the lines from the point to the angles of the quadrilateral shall be a minimum.

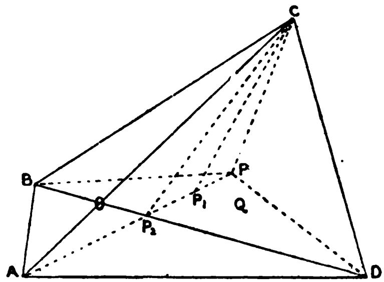


FIG. 14.

Here, assuming the student to have no knowledge of the property established in Ex. 4, the problem is not quite so simple. Let us see how it is to be dealt with.

Draw first the quadrilateral A B C D (Fig. 14), and from some assumed point, P, draw P A, P B, P C and P D. Then we have to inquire how to shift

P so as to lessen the sum of the distances P A, P B, P C, P D.

A very short inquiry suffices to show that we shall not gain much information by considering the lines P A, P B, P C, and P D in adjacent pairs. The inquiry might run somewhat in this way:—If P be brought towards B A, the sum of the lines P B, P A will diminish; but the sum of the lines P C and P D will increase. We have no obvious signs showing whether the diminution or increase is the greater. Therefore we are not tempted to continue this mode of inquiry.

Can we, then, by taking the lines in alternate pairs, diminish the sum of one pair without increasing the sum of the other? By bringing P towards the line B D (which we draw, at this point of the inquiry), the sum of the lines B P, P D is diminished (Euc. I., 21). Now if this were done without any attention to the lines P C, P A—for instance, if P moved to Q —it would not be easy to assert that the sum of the *our* distances from the angles was diminishing. But if P be made to move along P A, as to P_1, then—since C P_1 is less than P C and P P_1 together, we are diminishing, not merely the sum of the distances from B and D, but the sum of those from C and A. So long, then, as we continue this process, we cannot be going wrong. So that if we bring P to P_2—the intersection of P A and B D—we have diminished the sum of the distances as much as *this* process allows us to do. It is now obvious that by shifting our point

from P_2 towards A C, along the line P_2 B, we are yet farther diminishing the sum of the distances, *until* we reach the intersection of P_2 B and A C (which we here draw in). At this point of intersection, O, the second process has done all it can do for us. We see also that O is a fixed point within the quadrilateral, since it is the intersection of the diagonals. Also, P being *any* point, our process shows that wherever our point be taken, the sum of the distances diminishes continually as the point is made—by the double process above described—to approach O. Thus we are quite certain that O is the required point. Instead, however, of proving this by going through the necessary steps of the above process—which *would* be a sufficient proof—the student should give the proof in the following form, obviously suggested by the process he had before followed :—

Draw the diagonals A C, B D, meeting in O; then O is the required point. For, let P be any other point, and therefore not on *both* diagonals—say not on B D—then B P and P D are greater than B D (Euc. I., 20), and A P and P C are not less than A C (greater than A C if P do not lie in A C); hence P A, P B, P C, and P D, are together greater than O A, O B, O C, and O D together.

We have given the process determining the solution in the form which would most probably suggest itself. The double process is also very instructive and suggestive. But the practised geometrician would probably notice at once that the approach of P

towards O, *in a straight line*, diminishes at once the sum of P B, P D, and that of P A, P C. Hence we would argue, in presenting the proof, O *must* be the point we seek; for let any other point give a minimum sum, then, by taking a point nearer O, we obtain a less sum—that is, said point does *not* give a minimum: which is absurd.

The student must not always expect, however, to see so obvious a method of arriving at a maximum or minimum as in the preceding proposition. He must be ready to apply *tentative* methods. Take, for instance, the property established in the scholium to Ex. 8, and suppose we have the following problem:—

Ex. 10.—*Two points, P and Q (Fig. 15), lie on the same side of the line A B. It is required to find a point in A B such that the sum of its distances from P and Q shall be a minimum.*

We are supposed to know nothing of the property above mentioned. We might proceed then as follows:—

Take the two points at very unequal distances from A B.[1] Draw P D, Q E perpendiculars on A B. Then it is very obvious that the point we seek is not likely to lie outside D E. In order to *see* the sums of lines to D and E, produce P D to F, making D F equal to D Q and P E to G, making E G equal to E Q. Then it is obvious from the figure that P F is greater than P G; so that E *may* be the point

[1] In problems on maxima and minima it is very important that inequalities of this sort should be sufficiently marked.

we seek, but D certainly is not. But let us try intermediate points. Take C_1, and draw $P C_1 H$, making $C_1 H$ equal to $C_1 Q$. Then as drawn, $P C_1 H$ seems certainly not less than P G. Take C_2 nearer to E, and draw $P C_2 K$, taking $C_2 K$ equal to $C_2 Q$. We see that P K is obviously less than P G. Thus we learn that the point we seek lies between D and E but nearer to E than to D. If we were not restricted

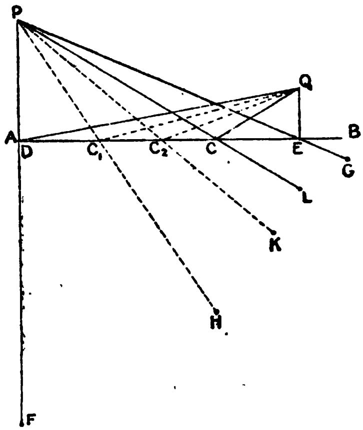


FIG. 15.

as to the books of Euclid we were free to make use of, we might be tempted to guess that the point we seek might lie at distances from D and E, proportional to P D and Q E. This, indeed, would lead to the same result as we shall proceed to by another method. But we suppose the student limited to the use of Book I. He considers, then, what determinate

point there can be in A B nearer to E than to D. He quickly rejects any points depending on the equi-division of the line. For instance, he cannot suppose that C E is *necessarily* a fourth part of D E; for since there is nothing to prevent P from being at the same distance as Q from A B, it is clearly not absolutely necessary that C E should be at all unequal to C D. The inequality depends on the inequality of P D and Q E, and may naturally be supposed to vary with the extent of the latter inequality. Our student can

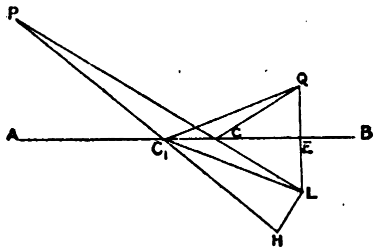


FIG. 16.

hardly fail, we think, to light on the supposition that C ought to be so taken that the angles P C D and Q C E should be equal. He would try this, drawing now a new figure, as follows:—

Draw a line A B (Fig. 16), and from a point, C, in it draw C P and C Q, inclined at equal angles to A B, *and unequal.* Take another point, C_1, and join P C_1, C_1 Q. Produce P C, P C_1, to L and H, making C L, C_1 H, equal respectively to C Q, C_1 Q. Then we have to prove that P C_1 H is greater than P L.

Join H L; then we should have to prove the angle

P L H greater than the angle P H L. This is obviously the case since the angle C_1 L H is equal to the angle C_1 H L (Euc. I., 5).

Or we might have noticed that the angle L C E is equal to the vertical angle P C A (Euc. I., 15) and therefore (*hyp.*) to Q C E. Also, C L is equal to C Q and the triangle L C Q (here we draw in Q E L) is isosceles, C E being the bisector of the angle contained by the equal sides. Hence C E is at right angles to Q L and bisects Q L in E. It is a very obvious consideration, at this point, that if we join C_1 L we shall have Q C_1 L an isosceles triangle, C_1 Q being clearly equal to C_1 L (Euc. I., 4). Hence P C_1 and C_1 L together are equal to P C_1 and C_1 Q together. But P C_1, C_1 L together are greater than P L (Euc. I., 20). Hence P C_1 and C_1 Q are together greater than P C, C Q together. We find, then, that our surmise is correct, for what we have proved for P C_1, C_1 Q_1 can be proved equally well wherever C_1 may be taken. Thus the problem is solved. It is not necessary to give the synthetical statement of our solution, since this has already been given in the scholium to Example 8.

It may be argued that such tentative processes as we began with here are not mathematics. To this it is to be answered—first, that the art of guessing well is an important aid to the mathematician; and secondly, that we deal with our guesses by means of mathematical reasoning, and thus gain all the benefit available from mathematical processes.

But further, there are no *laws* for applying simple geometry—that is geometry resembling Euclid's—to deductions; and therefore in many cases we have no choice but to make use of tentative methods.

VII. Non-Euclidian Devices.

We may remark in passing that there is no absolute necessity for restricting ourselves in all respects to Euclid's manner. Take as an instance his treatment of the famous *pons asinorum.* In dealing with this, as with all other propositions, he confines himself entirely to constructions which he has shown to be possible. Therefore, the following proof of the first part of the proposition would not be *in his manner*, though it would be difficult to find any flaw in the reasoning.

There must be *some* line which divides B A C (Fig. 17) into two equal angles.[1] Let A E represent this line. Then in the triangles B A E, C A E, B A is equal to A C (*hyp.*); A E is common; and the angle B A E is equal to the angle C A E. Therefore (by I., 4) the angle A B E is equal to the angle A C E.

Again, the following proof of both parts of the proposition is complete, though not in Euclid's manner:—

Conceive that the figure formed by the lines F K, F L, and G H (Fig. 18) is one that would coincide exactly with the figure formed by the lines A D, A E,

[1] The assumption here is precisely the same in character as that made in defining a right angle.

and B C; F K coinciding with A D (Fig. 18), F L with A E, and G H with B C. Now conceive the figure F K L to be turned face downwards, and so applied to the figure A D E that F K may coincide with A E; then since the angle G F H is equal to the angle C A B, F L coincides with A D. Also since A B, A C are equal to each other, and also to F G, F H, the points G and H coincide with the points C and B, and G H with C B. Thus the angle

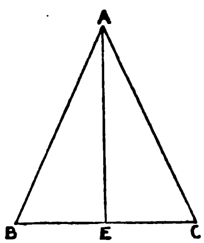


FIG. 17.

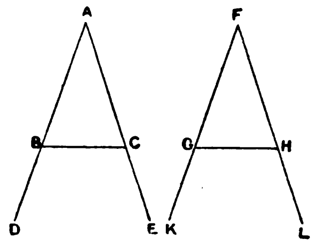


FIG. 18.

A B C coincides with and is equal to the angle F H G. But by our supposition, the angle A C B is equal to the angle F H G. Therefore the angle A B C is equal to the angle A C B. In like manner D B C coincides with G H L;[1] but, by our supposition, B C E is equal to G H L. Therefore D B C is equal to B C E.

Or, we may produce A B and A C in Fig. 17, and

[1] Here we assume as axiomatic the property which Simpson has attempted to prove in the corollary he has added to I., 12. He forgot, apparently, that Euclid had already (in Prop. 4 and elsewhere) assumed the property as self-evident, and that Prop. 12 itself cannot be solved on any other assumption.

conceive the part of the figure to the right of A E rotated round A E till it falls on the part to the left, and then show the perfect coincidence of the two portions.

In attacking geometrical deductions we are often compelled to assume in this way the existence of figures which are clearly *conceivable*, though we may not know precisely how to construct them, or though it may even be impossible to construct them by any of the ordinary geometrical processes. The following example of a problem in geometrical maxima and minima affords an instance:—

Ex. 11.—*A C B* (*Fig.* 19) *is part of a circle whose centre is at O. The points P and Q lie without the circle. Determine under what conditions the sum of the distances P C and Q C will be a minimum.*

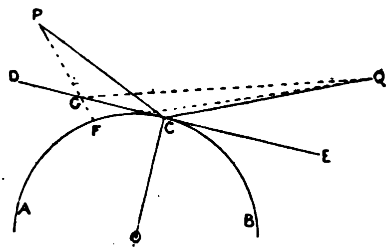


FIG. 19.

Here, guided by Examples 8, 9, to which the above is supposed to be given as a rider, we are readily led to the inference that P C and Q C should be equally inclined to the tangent at C. Now there is no simple method of determining C so that this

relation may hold. But it is clear that there must be *some* position of C for which it holds. Conceive, then, that P C and Q C are equally inclined to D C E, and let us inquire whether their sum is a minimum. Take any point F in A C, and join P F and Q F. Then we have to show that P F and Q F are together greater than P C, C Q. Let P F meet D C in G and join G Q. Then P G and G Q are together greater than P C and C Q (Example 9); and P F, F Q are clearly greater than P G, G Q (Euc. I., 20). Hence, *à fortiori*, P F, F Q are together greater than P C, C Q. Therefore the sum of P C and C Q is a minimum.

Cor.—Join C O; then the angle P C O is equal to the angle Q C O, and we may express the relation deduced above thus:—

The sum of the lines drawn from any point without a circle to a point on the circumference will be a minimum when the two lines are equally inclined to the radius drawn to the last-named point.

The subject of geometrical maxima and minima is a wide one, but we shall content ourselves here by adding three in which areas are dealt with.

Ex. 12.—*Two sides of a triangle being given, it is required to construct the triangle so that its area shall be a maximum.*

Let A B, B C (Fig. 20) be the lengths of the given sides.

With centre B and radius B C describe the circle C D F E. Then if we draw any radius B D or B E (these radii accidentally omitted from the figure should

be drawn by the student), and join A D or A E, it is clear that the triangle A B D or A B E thus constructed will have sides A B, B D, or A B, B E of the required length; and it is obvious that the area of any triangle thus formed will be greater or less according as the distance of its vertex from the line A B C is greater or less. We have not, indeed, any problem in Euclid which expressly states this as a truth respecting triangles on the same base, but the property is clearly involved in the proof of I., 39.

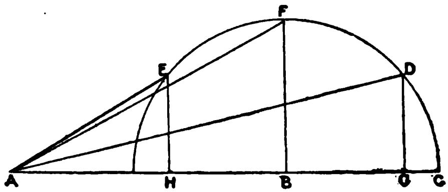


FIG. 20.

Now since the vertex must lie on the circle C D F E, it is obvious that the distance of the vertex from A B C can never exceed the radius of this circle, and can only be equal to the radius when the side adjacent to A B is at right angles to A B. Draw B F at right angles to A B, and join A F. Then the triangle A B F is the triangle of maximum area under the given conditions. The proof consists in showing that D G or E H drawn perpendicular to A B C is less than B F. This is evident; for in the right-angled triangle B D G, the angle D B G is less than a right angle; therefore D G is less than B D,—that is, than B F.

VIII. Perimeters of Triangles.

Let us next try a problem which is the converse of Ex. 5.

Ex. 13.—*To determine the greatest of all the triangles which can be constructed upon a given base and with a given perimeter.*

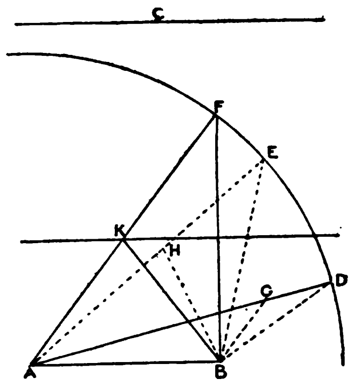


Fig. 21.

Let A B (Fig. 21) be the given base, C the sum of the remaining two sides.

Now, with a knowledge of the property established in Ex. 5, it is of course very easy to see what is the solution of our problem. But we shall assume that the student is dealing with the problem independently. With centre A and radius equal to C describe the arc D E F, and draw radii A D, A E, and A F. Then if from B we draw lines B G, B H,

B K in such a way that B G is equal to G D, B H to H E, and B K to K F, it is obvious that each of the triangles A G B, A H B, A K B has the required perimeter. Now it is an obvious consideration that if B G is equal to G D, the angle G B D is equal to the angle G D B (we here draw in B D), and, therefore, that in order to draw B G so as to be equal to G D, we have only to make the angle D B G equal to the angle G D B. So that having a *construction* for determining any number of triangles, it is presumable that we shall find materials for determining the triangle of maximum area. But first let us see if anything is suggested by an examination of the figure. We see first that the triangle gradually increases as the angle at A increases. But there is clearly a limit to this increase. For it is obvious that we might have taken B as the centre of a circle with radius C, and thus have shown that the triangle increases as the angle at B increases. We are led, therefore, at once to the consideration that our triangle will have its greatest area when the angles at A and B are equal.

To see whether this is the case, we construct a new figure (Fig. 22), in which we omit all unnecessary parts of the former figure, and draw A K F so that, when the triangle A K B is completed, the angle K A B shall be equal to the angle K B A. We then draw K L M parallel to A B, knowing that it is on the distance of this parallel from A B that the area of the triangle A K B depends. We take

A E pretty near to A F (seeing that the triangle has obviously a *nearly* maximum area when the angles at A and B are equal, so that any great departure from equality makes the triangle considerably smaller). Let A E intersect K L M in L. Then, if we can show that B H, drawn as before, falls between B A and B L, our surmise will have been proved to be correct. Now the angle H B E, by our construction, is equal to the angle H E B;

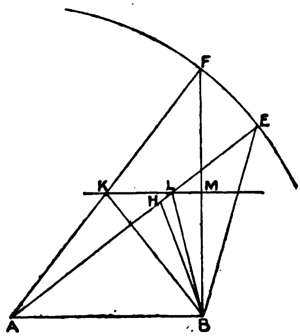


FIG. 22.

therefore we must show that the angle L B E is less than the angle L E B, or L E less than L B (Euc. I., 19); therefore, adding A L, we have to show that A E (or C) is less than A L, L B together. This is the problem dealt with in Example 5, and thus the rest of the work corresponds with the work in that example. We find that A L and L B are together greater than A E, so that H *does* fall below L; and the triangle A K B is greater than the triangle

A H B. Our surmise is, therefore, shown to be correct, and the problem is solved.

It will be noticed that a problem in maxima and minima loses a large part of its difficulty when, as is usually the case, we are merely asked to prove that such and such relations supply a maximum or a minimum. In the case of Ex. 13, indeed, inspection supplied a tolerably obvious solution; but this seldom happens. Presented in the usual form, the above problem would run.

Of all triangles on a given base, and having a given perimeter, the isosceles triangle is the greatest.

Thus given, the problem reduces immediately to the case of Ex. 5.

Ex. 13 fitly introduces the following, which belongs to a class often found perplexing:—

Ex. 14.—*Of all triangles having a given perimeter, the equilateral triangle is the greatest.*

The difficulty in a problem of this sort resides in the fact that we have three elements to consider, all of which admit of being changed. In Example 13 we only had two sides to consider, and when a length had been selected for one, the other was determined at the same time. In Example 14 we have three sides, and must assign lengths to two before the final condition of the triangle is determined. This would be found to afford no assistance towards the solution of the problem. The way to proceed is to assign a length to one side, provisionally, and then to consider what relation must hold between

the two remaining sides, whose sum is now assigned, in order that the triangle may be as large as possible. This we have learned already from Example 13. Those two sides must be equal. Hence, whatever side we suppose assigned, the remaining two must be equal to make the area of the triangle a maximum. Therefore, obviously, the triangle must be equilateral. The proof of this would run as follows :—

Let A B C (Fig. 23) be the triangle having the

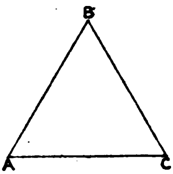


FIG. 23.

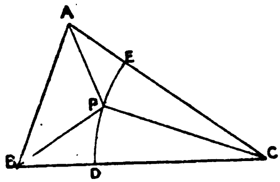


FIG. 24.

greatest possible area with a given perimeter. Then A B C must be the greatest possible triangle on a given base B C and with the sum of the remaining sides equal to the sum of B A and A C. Hence B A is equal to A C. But, also, A B C is the greatest triangle on the base A B with the given perimeter; hence, as before, A C is equal to B C. Therefore A B, B C, and C A are all equal.

As another instance of the application of this important method, we give the following :—

Ex. 15.—*A B C (Fig. 24) is an acute-angled triangle. It is required to determine the position of a point P within the triangle, such that the sum of the distances P A, P B, P C shall be a minimum.*

Assume P to be the required point. Then P A, P B, and P C together have a minimum value. Therefore, also, P A and P B have the least sum they can have *so long as the length of P C remains unchanged:* so that if we draw the arc D P E with radius C P and centre C, A P and P B are together less than the sum of any two lines which can be drawn from A and B to meet on the arc D P E. Hence (Ex. 11, Cor.) A P and P B are equally inclined to C P. Similarly A P and P C are equally inclined to B P. Hence the angles A P B, B P C, and C P A are all equal; and each, therefore, is one-third part of four right angles.

IX. Problems on Loci.

In Examples 12 and 13 we notice that, although the number of the triangles which can be constructed under the given conditions is infinite, yet all the triangles belong to a certain set or family. In Ex. 12, the vertices of all the triangles on the base A B lie on the circumference of the circle E F D. In Ex. 13 there is no curve along which the vertices are shown to lie; but if the reader were carefully to construct a number of triangles according to the method described in that example, he would find that the vertices all lie upon a certain curve, which, however, is not a circle.

These considerations introduce us to an important class of problems, called problems on *loci.*

If all points which satisfy certain relations can

be shown to lie on a certain line (straight or curved), and if every point on this line satisfies the given relations, the line is called the *locus* (or place) of such points.

A few examples will serve better than a formal statement to show (1), the nature of *plane loci*; (2), the nature of problems founded on them; and (3), the methods available for readily solving such problems. It must be premised that the complete solution of such problems requires that it should be shown that both the conditions stated in the above definition of a locus are fulfilled.

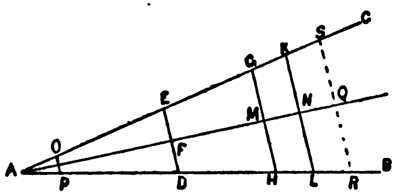


FIG. 25.

Ex. 16.—*The straight lines A B, A C (Fig. 25) intersect in A. From A equal parts A D and A E are cut off from A B, A C respectively. E D is bisected in F. Find the locus of all such points as F.*

Take A G equal to A H, A K equal to A L, and bisect G H in M, K L in N. Then it seems from the figure that the locus must be a straight line, whose direction is such as will carry it through A. A moment's consideration shows that the locus, whatever it be, must pass up to A; for if we conceive

equal lines, A O, A P, very small indeed, the bisection of O P will be very near indeed to A. Again it will occur, from a consideration of the figure, that the locus is a straight line bisecting the angle A. Now, assuming for the moment that A F M N is such a line, we see that the triangles A N L, A N K are equal in every respect (Euc. I., 4), and this leads us at once to the proof we require. For, because the base, K L, of the isosceles triangle A K L is bisected in N, therefore N lies on the bisector of the angle K A L. Similarly every point obtained in accordance with the given conditions lies on the bisector of the angle K A L. It is clear, also, that every point in the bisector of the angle K A L fulfils the required conditions. For, let Q be such a point, and draw S Q R at right angles to A Q; then the triangles A Q S and A Q R are equal in every respect. (Euc. I., 26.) Therefore, A S is equal to A R, and S Q to S R; that is, Q is a point fulfilling the required conditions.

Points in the production of Q A beyond A cannot be said to fulfil the requisite conditions, because nothing has been said of the production of B A and C A beyond A.

Ex. 17.—*Determine the locus of the vertices of all the triangles which stand upon a given base and have a given vertical angle.*

Let A B (Fig. 26) be the given base, C the given angle.

Draw from A straight lines, A D, A E, A F, and from B draw B G, B H, B K, to make with A D,

A E, A F, respectively, the angles B G A, B H A, and B K A equal to the angle C.[1]

We see at once that G, H, and K do not lie in a straight line, so that we gather that the locus is circular, since loci of other figures are not dealt with in deductions from Euclid.

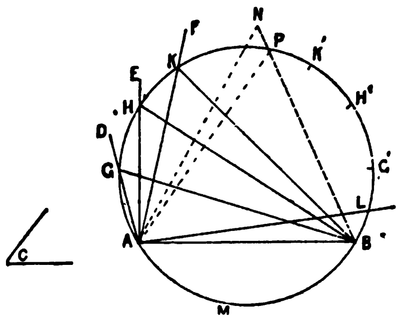


FIG. 26.

Now we notice that we might have drawn our lines from B instead of A, and that therefore the locus must have points, G′, H′, K′, situated in the same manner with respect to B as G, H, and K with respect to A.

It is already clear that a circle passing through or

[1] There is no problem in Euclid which shows us how to do this, but of course there is no difficulty in the matter. Among the subsidiary problems mentioned in the first part, one should be given showing how to draw a straight line in the manner required. Here, however, we do not require the problem at all; since we are dealing with the practical construction of the figure —about which there is no difficulty—not with the mathematical treatment of the problem.

near to A and B contains all the vertices. We see also that the circle cannot but pass *through* A and B, for if we draw A L very near indeed to A B, then B L drawn so as to make the angle B L A equal to C will clearly meet A L in a point very near indeed to B. We describe, then, a circle through A and B, and also (of course the circle is drawn by hand) through the points G, H, K, &c.

At this point we cannot fail to be reminded of III., 21, which tells us that all the angles in the same segment of a circle are equal. We see, therefore, that our surmise is correct, and that the circular segment on A B, containing an angle equal to the angle C, is the locus we require. All the points on this segment fulfil the required condition; but points on the remaining segment, A M B, do not do so. If triangles are to be drawn on one side only of A B, the segment A K B contains *all* the required points. For if any point, N, without the segment, fulfil the given condition, join N A and N B; let N B cut the segment A K B in P, and join A P. Then the angle A N B is equal to C (*hyp.*), but the angle A P B is equal to C (Euc. III., 21). Therefore the angle A P B is equal to the angle A N B, the greater (Euc. I., 16) to the less, which is absurd. In like manner, no point within the segment fulfils the required condition. Therefore, the segment A K B is the required locus.

Let us next try the following problem :—

Ex. 18.—*Determine the locus of the middle points*

of all the chords of a circle which pass through a fixed point.

The fixed point may be either within or without the circle. In nearly all cases of this sort it is well to begin with a point within the circle, trusting to the result thus obtained to guide us in the case of a point without the circle.

Let P (Fig. 27) be a point within the circle A B C D. We are to draw chords through P, and to bisect them. Draw, first, the diameter A P E C

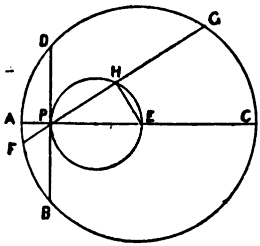


FIG. 27.

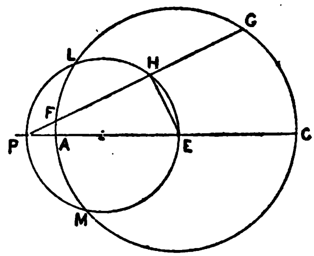


FIG. 28.

through P. Its bisection, E, is the centre of the circle. This is one point of the required locus. Draw next the chord B P D at right angles to A C. Then the point P is itself the bisection of D B (Euc. III., 3). Therefore P is a point on the required locus. Next draw a chord F P H G through P, and bisect in H. Then H, a point on the locus, is clearly not in the straight line joining P E, so that the locus is not a straight line. It is therefore probably a circle. Now we see at once that for every point we

get above A C there must be a corresponding point below A C. We see, then, the probability that the required locus is a circle of which P E is the diameter. But even if the student failed to see this at once, he would readily detect it when he had drawn several more chords through P (above and below P C) and bisected them. We describe, therefore, a circle E H P, of which we assume P E to be the diameter, and we look for a proof that a chord drawn as F P H G *would* be bisected in H, where it meets the circle thus drawn. It will clearly be well to join E H. When this is done, one of two well-known properties can hardly fail to occur to our mind. We might either remember that the angle in a semicircle being a right angle, E H will be at right angles to F G, if P H E really is a semicircle; or we might remember that the line from the centre of a circle to the bisection of any chord is at right angles to the chord, so that the angle E H P is a right angle independently of any consideration of the assumed circle P H E. Of course, if we thought of the first property we should be led immediately to the second, and *vice versâ*. The two properties are, in fact, interdependent; and we see at once that their interdependence involves the solution of our problem.

We now write out the solution in the following form :—

Let A B C D be the given circle, P the given point.

First, let P lie within the circle. Draw any chord

F P H G, and bisect F G in H. Find E, the centre of the circle A B C D, and join E H. Then E H is at right angles to F G (Euc. III., 3); therefore H is a point on the circle of which P E is a diameter (Euc. III., 31). But F G is any chord through P. Therefore the bisections of all such chords lie on the circle E H P. Also it is clear that every point on this circle bisects some chord through P. Therefore this circle is the locus required.

Next, let P lie without the circle (Fig. 28). Then the proof is the same[1] up to the words 'therefore the bisections of all chords through P lie on the circle E H P'; then we proceed:—It is also clear that points on the arc L E M bisect chords through P; and also that *every* point on this arc bisects *some* chord through P.

A readiness in determining the loci corresponding to different conditions will often be found serviceable to the student engaged in solving problems of different classes.

Suppose, for instance, that the following problem is set:—

Ex. 19.—*Let A, B, C (Fig. 29) be three given points, D a given straight line. It is required to find a point which shall be equidistant from the points A and B, and at a distance from C equal to the line D.*

[1] It is important to notice that in such a case as the above, *by putting the same letters at corresponding points in both figures*, the proof of one case may nearly always be made to apply to the other, either without change, or with such obvious changes as the student can have no difficulty in making.

In order that the distance of the point from C may be equal to the line D, it is clearly necessary that the point should lie somewhere on the circumference of the circle described with centre C, and radius equal to D. Let G E F be this circle.

Next, we inquire whether there is any *locus* containing all points equidistant from A and B. We join A B and bisect in H, giving one point H, clearly belonging to such a locus. Next, either by applying

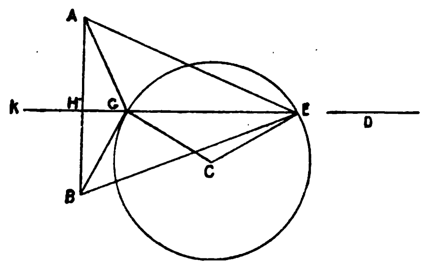


FIG. 29.

tentative methods, as in the above instances, or by the consideration of a few obvious facts, we find that the indefinite line K H G E, drawn through H at right angles to A B, contains all points equidistant from A and B. The line K H G E does not *necessarily* intersect the circle F G E. If it intersects that circle in two points, G and E, it is clear that each of these points satisfies the required conditions. For C G is equal to D (*const.*), and G A is equal to G B (Euc. I., 4. See also Ex. 1). Also, C E is equal to D and

E A to E B. If K H G E touch the circle there is only one point satisfying the given conditions. And clearly, if K H G E do not meet the circle, there is *no* point satisfying the given conditions. For if there were such a point, it would be at a distance D from C, and therefore would lie on the circle F G E. Also, it would be equidistant from the points A and B, and therefore would lie on K H E. In other words, the circle F G E *would* have a point in common with the line K H G E, which we have supposed not to be the case.

Let us consider the method applied in our last. One condition shows us that the point we seek *must* lie on a certain curve; another condition shows us that the point *must* lie on another curve. Therefore, the point we seek must lie at some intersection of the two curves. If there are more intersections than one, the problem has more solutions than one; if there is but one intersection, there is but one solution; if, lastly, the curves do not intersect, the problem is insoluble.

Let us take, as another instance, the following problem:—

Ex. 20.—*Let A B (Fig. 30) be a given straight line, C a given angle, D a given point within the given circle E F G. It is required to determine a point at which A B shall subtend an angle equal to the angle C, and which (point) shall be the bisection of a chord through D to the circle E F G.*

In order that A B may subtend an angle equal to

C at the required point, this point must lie, we find (as in Ex. 17), on the arc A E B, containing an angle equal to the angle C.

Again, in order that the required point may be the bisection of a chord through D to the circle E F G, this point must lie, we find (as in Ex. 18), on the circle L K M, which has for diameter the line joining D with K, the centre of the circle E F G.

These two loci—viz. the arc A E B and the

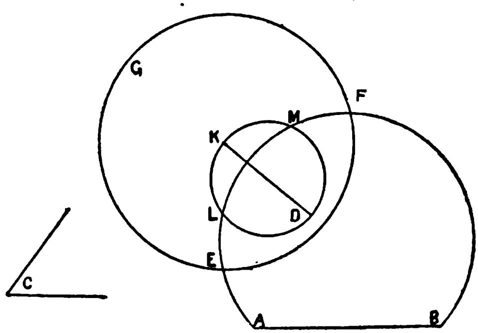


FIG. 30.

circle L K M—determine by their intersection the points which satisfy the required conditions. There may be two points, as in the case illustrated by our figure; or one point, if the circle L K M *touch* the arc A E B; or the two loci may not intersect, in which case the problem does not admit of solution.

We have supposed that the point is required to lie above A B. If not, then an arc equal in all respects to A E B, but applied on the opposite side of

A B, would include other points satisfying the first condition of our problem. It might happen that the circle L M K intersected the latter arc, instead of, or as well as, the arc A E B. Such point or points of intersection would also supply a solution of the problem.

Problems in maxima and minima also involve very frequently the discussion of loci.

Suppose, for instance, that the following problem is given :—

Ex. 21.—*A, B, C, and D (Fig. 31) are four fixed points. It is required to determine a point equidistant*

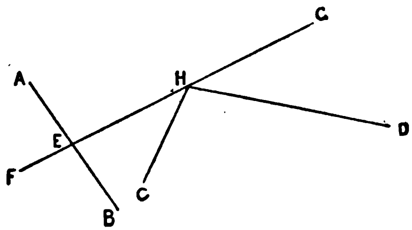


FIG. 31.

from A and B, and such that the sum of its distances from C and D shall be a minimum.

In this case we first find the locus of points equidistant from A and B. This, as in Ex. 18, is the line F G drawn at right angles to the line A B, through its bisection E (Fig. 31). We have, then, to find a point in F G such that the sum of its distances from C and D may be a minimum. We find (as in Ex. 11) that the point must be so taken—as at H—

that the lines from C and D to it shall make equal angles (C H F and D H G) with the line F G.

To take another simple instance, suppose we had the following problem :—

Ex. 22.—*A triangle is constructed on a given base A B (Fig. 32), and with a vertical angle equal to the angle C; to determine its figure that its area may be a maximum.*

Here we first inquire what is the locus of the vertices of all the triangles which can be constructed

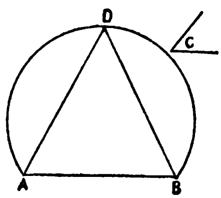


FIG. 32.

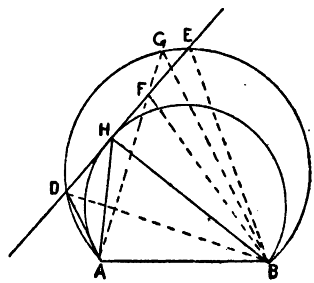


FIG. 33.

on the base A B with a vertical angle equal to the angle C. We find, as in Ex. 17, that the locus is the arc A D B, containing an angle equal to the angle C.

After this we find no difficulty in determining the triangle of maximum area. The vertex must clearly lie at that point of the arc A D B which is farthest from A B; and D, the bisection of the arc, is obviously the required vertex. The student will at once *see* this; but perhaps he may find a little

difficulty in proving it. We leave this part of the problem to him as an exercise, having already examined the treatment of problems of this class. We note, however, that what he has to do is to show that a parallel to A B through D is farther from A B than the parallel through a vertex of any other triangle fulfilling the required conditions; and this will be established if it be shown that the parallel to A B through D is a tangent to the arc A D B.

Sometimes a familiarity with the treatment of problems on loci serves us in a somewhat more subtle manner, as in the following problem :—

Ex. 23.—*A B (Fig. 33) is a given finite straight line. It is required to show where a point must be taken in the given indefinite line D E, in order that the angle subtended by A B from the point may be a maximum.*

Suppose we take any point, D, at random, in D E, and draw the lines D A and D B. Then, in inquiring whether the angle A D B is a maximum or not, it would be an obvious consideration that the segment of a circle, A D E B, described on A B, contains all the points from which A B subtends an angle equal to the angle A D B. From the point E, therefore, A B subtends an angle, A E B, equal to the angle A D B; and from any point, F, between D and E, it is clear that A B subtends an angle greater than A D B. For, producing A F to meet the arc A D B in G, and joining G B, we see that A F B is greater than A G B (Euc. I., 16),—that is, than A D B (Euc.

III., 27). It is clear, therefore, that we cannot have a maximum so long as the arc described on A B, to pass through the particular point selected in D E, cuts D E in another point. Hence we arrive immediately at the solution of our problem—viz. that the required point, H, is so situated that the arc on A B through H touches the straight line D E.

It is easy to draw a circle through two given points to touch a given straight line. Thus, let B A E D produced meet in T; then take T H so that the square on T H is equal to the rectangle T A, T B; a circle through B, A, and the point H will be the circle required. But, strictly speaking, the solution of the above problem is complete without the construction of the circle A H B, since we have assigned a *sufficient* condition for the determination of the required point in D E.

The consideration of problems on loci leads us to another class—or rather to two other classes of deductions—viz. those in which it is required to prove either that certain straight lines pass through one point, or that certain points (more than two) lie in a straight line.

X. Intersection Problems.

Such problems as I mentioned in the last lesson usually belong to a more advanced stage of study than that for which these simple papers are intended. They also often require the use of the Sixth Book.

It will suffice here to consider a few of the simplest cases.

Suppose we have such a problem as this given :—

The sides of the triangle A B C (Fig. 34) *are bisected in the points* a, b, c, *and the three straight lines* a k, b l, *and* c m *are drawn at right angles to B C, A C, and A B respectively : show that these three straight lines,* a k, b l, *and* c m *pass through a point.*

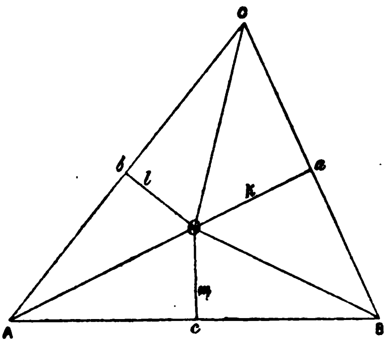


FIG. 34.

Here the student might at once refer to the fourth book, and find a proof in the circumstance that *a k* and *b l* have there been shown to meet at the centre of the circle through the points A, B, C. So also by the same book do the lines *a k* and *c m* meet at the centre of the circle through the points A, B, C. Now there is but one circle passing through these points ; for if there were two, two circles would intersect in three points, which is impossible. Hence *a k*, *b l*, and *c m* pass through the same point.

But, although this proof is sound enough, it is not independent, as a proof of this sort should be. Yet an actual and sufficient proof will run closely, as might be expected, on the lines followed in Book IV.

It is hardly necessary to say that the proof must be indirect. We can show, as in Book IV., that if *a k* and *b l* meet in O, the lines O A, O B, and O C are all equal. Then since A O = O B, a line

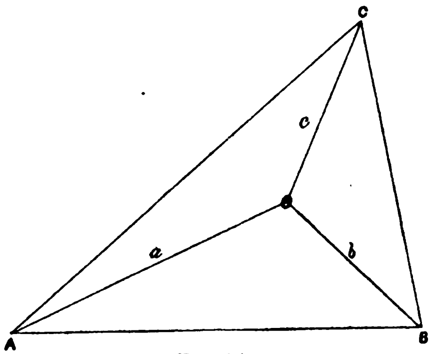


FIG. 35.

from O perpendicular to A B must bisect A B, in other words, must pass through *c*, and coincide with *c m*. Hence if we wished to put the proof in Euclidian form, we might begin by saying, If possible let *c m* not pass through the point O in which *a k* and *b l* intersect, but have some other position as *c m o* (not shown in fig.). Then after proving that A O = O B, we could show that O *c* is at right angles to A B. But *c m o* is at right angles to A B.

Wherefore from the same point *c* there can be drawn two straight lines at right angles to A B and on the same side of it—which is impossible, since all right angles are equal. Therefore the line through *c* at right angles to A B cannot lie otherwise than through O.

In a similar way we can deal with the problem :—

If the three angles of the triangle A B C (Fig. 35) *be bisected by the lines A* a, *B* b, *and C* c, *these straight lines will all pass through one point.*

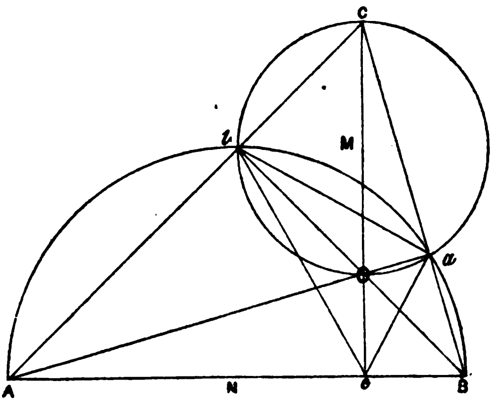


Fig. 36.

But now suppose we have this problem :—

From the angles A, B, and C, of the triangle A B C (Fig. 36) *lines are drawn at right angles to B C, C A, and A B respectively. These three straight lines will all pass through one point.*

Here again the indirect method must be employed. We may draw A *a*, B *b*, at right angles to B C, C A

respectively, and intersecting in O; then if we can prove that C O (produced if necessary) is at right angles to A B, what is required is done.

We have in this case the angles at b and a right angles; and it is nearly always well to try in such cases whether any good comes from noting that the angle in a semicircle is a right angle. This at once shows that a circle on O C as diameter will pass through $b\,a$; as will also a circle on A B as diameter. Suppose these circles drawn; or if any difficulty arises from the effort to *conceive* them as drawn, draw them in as in the figure.

Also it will obviously be convenient to draw in the lines $a\,b$, $b\,c$, $c\,a$.

We have now to show that C O c is at right angles to A B. If this be so, the angles c C A and c A C together make up a right angle, or are complementary to each other. Of these the angle c A C is a known angle; so that if we look for an angle *known* to be complementary to c A C, we may be able to prove that so also is c C A. Now the angle A B b is complementary to c A C by the construction. Can we show that $\angle \mathrm{AB}b = \angle c\mathrm{CA}$? We must try our circles. We see that $\angle \mathrm{AB}b = \angle\, b\,a\,\mathrm{A}$ on the same segment A b; and we see that $\angle\, b\,a\,\mathrm{A}$ or $b\,a\,\mathrm{O} = \angle\, b\,\mathrm{CO}$ on the same segment b O. This clearly serves our purpose. For we have

$$\angle\, b\,\mathrm{CO} = \angle\, b\,a\,\mathrm{O} = \angle\, b\,\mathrm{BA} = \text{compt. of CAB},$$

wherefore the angle C c A is a right angle.

XI. Problems about Shape.

At present there remains only one class of deductions to deal with—viz. those in which questions of *shape* are involved. There are many problems which, although sufficiently simple and easy, do not admit of being solved without a reference to the sixth book of Euclid; and there are others which are much more readily solved by means of the sixth book than without its aid.

Consider, for instance, the following example:—

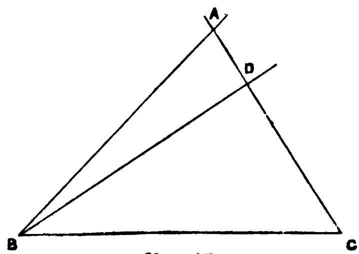


Fig. 37.

Ex. 24.—*A B C (Fig. 37) is a given angle; it is required to draw a line, B D, so that when through any point D in B D a line, A D C, is drawn at right angles to B D, D C shall be equal to three times A D.*

Here is a problem clearly depending on *shape*—for instance, not on the *length* of B D or A C. We see that if we can divide any angle equal to A B C in the required manner our problem is solved; or, rather, we have to construct a figure resembling A B C D as A B C D is *supposed* to be drawn.

We notice that A D is one-fourth of A C, and D B at right angles to A C. We therefore draw a straight line E F (Fig. 38), take H F equal to one-fourth of E F, and draw H G at right angles to E F. All that is now required is that we should determine G so that the angle E G F may be equal to the angle A B C. This is readily effected, since we know how to describe on E F an arc E G F containing an angle equal to the angle A B C (Euc. III., 33); the intersection of the line H G with the arc E G F gives us the required

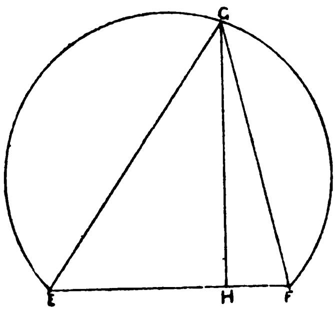


FIG. 38.

point G. We join E G, G F; then the angle E G F is equal to the angle A B C.

Now if we draw B D so that the angle A B D is equal to the angle H G F, then the remaining angle D B C is equal to the remaining angle H G E. Through any point D in the line B D thus obtained draw A D C at right angles to B D. Then obviously the triangle A B D is similar to the triangle G H F; therefore A D is to B D as F H to H G. Similarly

B D is to D C as G H to H E; therefore, *ex æquali*, A D is to D C as H F to H E. But H F is one-fourth of E F; therefore H E is equal to three times H F; and hence D C is equal to three times A D.—Q.E.F.

We do not give the considerations which lead to the last lines of the proof. The considerations respecting shape which led to the construction may be looked on as obvious, although (as is often the case) it may not be quite so obvious how the *proof* is to be made to depend on properties established in the sixth book.

In a problem of the above type we cannot well avoid the use of the sixth book. I now give a problem which can be solved by the third book, but one can scarcely doubt that the solution depending on the sixth book is that which would *naturally* occur to a person dealing with the problem as a new one:—

Ex. 35.—*Let A B, A C (Fig. 39) be two straight lines meeting in A, D a given point. It is required to draw a circle which shall pass through the point D and touch the lines A B, A C.*

We first notice that the circle must have its centre in the line A E, which bisects the angle B A C. For, taking any point, F, on this bisector, and drawing perpendiculars F H and F G on A B and A C respectively, we see that F H is equal to F G (since the triangles F A H, F A G are equal in all respects, Euc. I., 26). We describe a circle H G K, with centre F and distance F H or F G, and touching A B and A C in H and G (Euc. III., 16). But this circle

does not pass through D. It is obvious, however, that if we draw A K D, cutting the circle H G K in K, then the figure formed by the lines A H, A G, the point K, and the circle H G K exactly *resembles* that which will be formed when our problem is solved. If, then, we can only form a figure resembling that we have constructed, but such that the circle shall pass through D, the problem will be solved. Now we see that F lies in a defined direction with respect

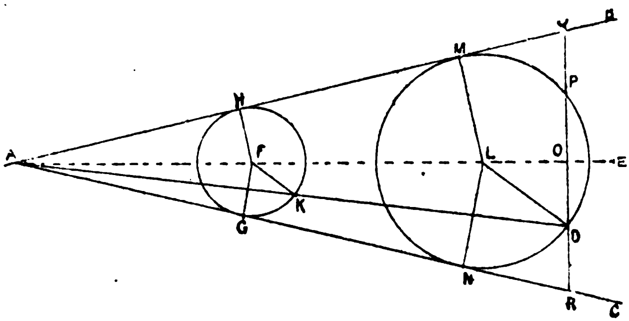


FIG. 39.

to K—in other words, the angle A K F does not vary with the size of the circle drawn as H G K was drawn. We have then only to draw D L so that the angle A D L is equal to the angle A K F—that is, we have only to draw D L parallel to K F, to determine L, the centre of the circle we require. The proof runs thus:—

Draw L M and L N perpendicular to A B and A C respectively. Then, from the similar triangles

A L D, A F K, L D is to A L as F K to A F. Again, from the similar triangles, A L N, A F G, A L is to L N as A F to F G. Therefore, *ex æquali*, L D is to L N as F K to F G. But F K is equal to F G; therefore L D is equal to L N—that is, to L M. Hence a circle described with centre L at distance L D will pass through M and N, and touch A B and A C in these points respectively, since the angles at M and N are right angles.

NOTE.—Of course there is no difficulty in solving this problem without the aid of any book beyond the third. The obvious course of proceeding is by way of analysis. Let D N M be the required circle, having its centre at L. A L E is the bisector of the angle B A C and can be drawn at once. R D O P Q at right angles to A E can also be drawn, and gives P, a point on the required circle; for D O = O P. Then one can hardly miss the relation that the square on R N is equal to the rectangle under R D, R P; whence N is given, and the required circle, passing through the three known points, P, D, N, is determined.—Q.E.F.

SECTION II.

NOTES ON EUCLID.

WITH SPECIAL REFERENCE TO THE SOLUTION OF GEOMETRICAL PROBLEMS.

I HAVE often wondered that among the various attempts to correct the obvious defects—for educational purposes—of Euclid regarded as a text-book of mathematics, nothing should have yet been done to remodel the book itself. Various writers have published books of geometry, each fondly hoping that his book will not only displace Euclid, but dispose of all rivals. The result has proved rather confusing. A boy who has been at a public school where one of these books has been used, goes perhaps afterwards to a private tutor who prefers another text-book of geometry; thence, perhaps, to a London college, where yet another book is employed; and finally to one of the Universities, where he finds Euclid still holding the place of honour.

Now, if Euclid simplified could be put into boys' hands at school, and all other text-books diligently

eschewed, there would be a common system at all schools, and no trouble when the old-fashioned Euclid was taken up, at whatever stage of mathematical progress.

There can be no doubt Euclid perplexes many boys, and disgusts not a few. For my own part, though I was introduced to Euclid in the absurdest of ways, I loved him from the beginning. I had been counted rather a dullard at Geometrical Exercises, which I disliked, because there were Rules without Reasoning. Just as the foolish arithmetics of those days told us in hard words what to do, but never showed us why, so the Geometrical books told us to rule this line and describe that circle, in order to bisect lines, set up perpendiculars, and so forth, with no proof that the methods were sound. Besides, while mapping (a most instructive exercise), I had intuitively invented such methods for myself; so that rules had not even the charm of novelty. At this stage of my progress— or want of it—a preposterous under-master pitchforked me into a higher class where Euclid was read, and where, as it chanced, the 16th Proposition of the First Book was in hand. It was a new thing to me to find reasoning about matters geometrical. Theoretically, the folly of putting me at the 16th Proposition *first* ought to have made Euclid hateful to me; but, as a matter of fact, the case proved otherwise: I loved him from the first. I read alone the Definitions, Axioms, and preceding propositions; then went along with propositions ahead; till, before very long, I was in the

Spider's Web of the last proposition but one of Book XII. Yet I was by no means a sound, only an eager, student of Geometry; for I remember devising a new construction for that most delightful of all propositions the 10th of Book IV., which, though much shorter and easier than the original, laboured under the trifling defect of being incorrect.

Still, I think my case was exceptional. Most boys do not take kindly to Euclid, and certainly there is much in his outer appearance which is not inviting. In particular, the method of first giving the abstract proposition, and then describing a particular case, tests somewhat painfully the young student's power of attention. It is so much harder to make out what the enunciation means than it would be if each part were explained as in the opening words of the proposition, that we cannot wonder if boys are bewildered and wearied. To the more advanced it is pleasing to note how each enunciation has the qualities of a good definition in precisely indicating the abstract idea without any reference to a special case. But Euclid did not write for boys.

Again, there is a charm in the skill with which Euclid, having adopted a certain method, gets over the difficulties involved in applying that method to particular cases. The famous *Pons Asinorum* is a case in point. Euclid's plan will not allow him to use the bisector of the angle B A C, because he has not yet shown how that bisector can be drawn. Nor can he allow himself to suppose his initial figure,

repeated line for line, and then applied, after being turned over, to the original figure, after the manner already employed in Proposition IV., because he has not yet shown how the 'copy' is to be made. Either method would have given him a very simple proof, and as it is certain that there *must* be a line bisecting the angle B A C, and again that another figure precisely like that already drawn is conceivable (in the same sense that a straight line or a circle is conceivable from its definition), he was, logically, free to employ either plan. But he had assigned himself certain limits, and he makes out his proof within those limits very ingeniously and prettily—though confusingly to many boys.

The First Book of Euclid treats chiefly of the properties of triangles and parallelograms. An examination of the book suffices to show that Euclid had proposed a definite line of treatment leading up to certain important propositions. Hence many useful properties are left untouched in this book. It is surprising, however, how many valuable propositions Euclid has succeeded—by a judicious method of treatment—in introducing into his plan without marring its symmetry.

In attacking deductions either immediately depending on the First Book of Euclid or involving it in part only, it is necessary that the student should have at his fingers' ends, so to speak, all the most useful properties established in the First Book, and also several important properties deducible from this

book. We proceed to examine the most valuable of these.

In the first place, let us run through the First Book and notice whether there are any properties whose converse theorems, though not proved in Euclid, may be readily established.

Euclid has proved the converse of Prop. 5 in Prop. 6, of Prop. 13 in Prop. 14, of Prop. 18 in Prop. 19, of Prop. 24 in Prop. 25, of Props. 27 and 28 in Prop. 29, of Prop. 37 in Prop. 39, of Prop. 38 in Prop. 40, and of Prop. 47 in Prop. 48. The other propositions which admit of a converse are the following :—

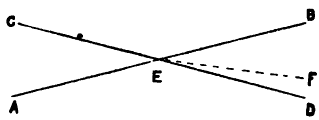


FIG. 40.

Euc. I., 15, of which the converse is, 'If two straight lines C E, D E (Fig. 40), on opposite sides of a line A B, make equal angles C E A, D E B with A B; then C E and E D are in the same straight line. This is obviously true, since if C E produced fell in some other direction, as E F, we should have the angle B E F equal to the vertical angle C E A, and therefore to the angle B E D, which is absurd. We may refer to this proposition as Euc. Book I., Prop. 15, *Conv.*

Euc. I., 17.—The converse of Prop. 17 is Axiom 12. We touch here on the great defect of Book I.,

a defect, however, with which our subject does not lead us to deal.

Euc. I., 34.—This proposition contains three theorems, each of which has a converse, but the converse of the third is not true. The converse of the first part of the proposition is this:—*If the opposite sides of a four-sided rectilinear figure are equal, the figure is a parallelogram.* This is obviously true; for, having A B (Fig. 41) equal to D C and A D equal to B C, also the base B D common, we have the angle B A D equal to the angle B C D (Euc. I., 8), and the triangles B A D, B C D equal in all respects (Euc. I.,

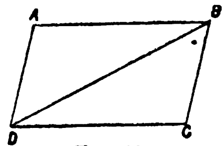


FIG. 41.

4), so that, the angle A B D being equal to the angle B D C, A B is parallel to D C; and similarly, A D is parallel to B C. We may refer to this proposition as Euc. I., 34, i. *Conv.*

The converse to the second part is also true. It is—*If the opposite angles of a four-sided rectilinear figure are equal, the figure is a parallelogram.* In this case, having the angle D A B (Fig. 42) equal to D C B, and A B C to A D C, we have the sum of the angles D A B and A B C equal to the sum of the angles A D C and D C B, or either sum equal to half the sum of the four interior angles of the figure—

that is, to two right angles (Euc. I., 32, Cor. 1): hence (Euc. I., 28) A D is parallel to B C, and similarly A B is parallel to D C. We may refer to this proposition as Euc. I., 34, ii. *Conv.*

The converse of the third part of the proposition is not a true theorem, for it is clear that besides D C B (Fig. 41) an infinite number of triangles may be drawn on the base B D equal to A B D, each of which would give a quadrilateral divided by B D into two equal triangles, but this quadrilateral would not be a parallelogram.

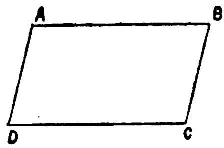


FIG. 42.

Euc. I., 35 and 36.—Each of these propositions has a converse which may be established in the manner of Props. 39 and 40. We may refer to these converse theorems as Euc. I., 35 and 36, *Conv.*

Euc. I., 43 has the following converse:—*If H K G (Fig. 43) is drawn parallel to the two sides A B, D C of a parallelogram A B C D and E K F parallel to the other two sides, in such a manner that the parallelogram H F is equal to E G, then H F and E G are complements about the diagonal A C—in other words, the point K lies on the diagonal A C.* This may readily be proved to be true. For, if A C had some other position, as A L C, then, drawing M L N parallel to

A D or B C, we have the complement B L equal to the complement L D: therefore B K is less than L D;

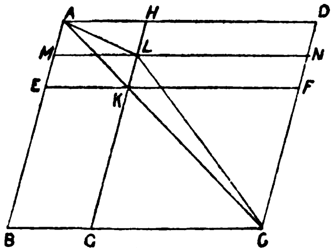


Fig. 43.

but B K is equal to K D; therefore K D is less than L D—the whole less than a part, which is absurd. Hence B K and K D are complementary parallelograms, or K is a point in the diagonal A C. We may refer to this proposition as Euc. I., 43, *Conv.*

Let us next examine the particular objects which Euclid appears to have had in view in the First Book, and see whether any additions seem to be suggested, noting at the same time those propositions which are of frequent use to the geometrician.

The first proposition shows us how to construct an equilateral triangle. The same method is clearly applicable to the construction of an isosceles triangle on any finite line.

The student is not likely to neglect the application of Prop. 3 (to which Prop. 2 is wholly subsidiary).

Prop. 4 is the first determining the equality of triangles. The others are Props. 8 and 26. We

learn from them that the equality (i) of two sides and the included angle; (ii) of the three sides; (iii) of two angles and a side opposite to equal angles in each triangle; and (iv) of two angles and a side adjacent to them, suffices to determine the equality of two triangles *in all respects.* For although Euclid limits the proof in Prop. 8 to the angles contained by the *sides* (as distinguished from the angles contained between the base and either side), yet, since any side might have been taken for base, the equality of each angle of one triangle to the corresponding angle of the other is established.

Now there are six elements in the determination of a triangle—the three angles and the three sides. It will appear, on consideration, that Euclid has combined these, three and three, in four ways out of six possible ways. It remains, then, only to consider the remaining two.

It is obvious that if the three angles of one triangle are equal to the three angles of another, the triangles are not necessarily equal. For it follows from Euc. I., 29, that if we draw a parallel to one side of a triangle, either within the triangle or else to meet the other two sides produced, we form another triangle, unequal to the first, but having equal angles.

There remains only the case of two triangles having two sides of the one equal to two sides of the other each to each, and an angle opposite one side of one triangle equal to the angle opposite to the equal

side of the other. In this case the two triangles are not *necessarily* equal. We may form the following proposition, which is an important one, as are also its corollaries.

PROP. I.—*If two triangles have two sides of the one equal to two sides of the other, each to each, and the angles opposite to a pair of equal sides equal; then if the angles opposite the remaining sides be both acute, or both obtuse, or if one of them is a right angle, the two triangles are equal in every respect.*

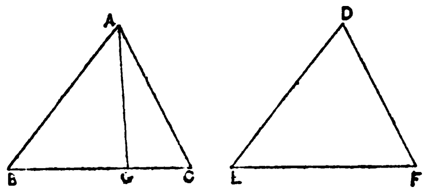


FIG. 44.

In the two triangles A B C, D E F, let A B be equal to D E, and A C to D F; also let the angle B be equal to the angle E.

First let the remaining angles C and F be acute.

If the angle A be not equal to the angle D, make the angle B A G equal to the angle D. Then the triangles A B G and D E F are equal in all respects (Euc. I., 26), therefore A G is equal to D F, and the angle A G B to the angle F. But since D F is equal to A C, A G is equal to A C, and the angle A G C to the angle A C G; hence A G C is an acute angle and A G B obtuse (Euc. I., 13). Therefore the

angle A G B is *not* equal to the angle F; which is absurd. Therefore the angle B A C is not unequal to D; that is, these angles are equal, and (Euc. I., 4) the triangles A B C and D E F are equal in all respects.

If the angles C and F are both obtuse, the proof is similar to the preceding; or, if we please, we may adopt a proof resembling that of the following case.

If the angle C is a right angle, we proceed as before until we have proved that the angle A G C is equal to the angle A C G. Thus we have two angles of the triangle A G C equal to two right angles, which is impossible. Therefore, as before, the triangles are equal in all respects.

COR. 1.—If the equal angles B and E are right or obtuse, the other angles are necessarily acute and the triangles are equal in all respects.

COR. 2.—If the equal sides A C, D F, are greater than the equal sides A B, D E, the angles C and F are necessarily acute (Euc. I., 18, 17). Hence in this case also the triangles are equal in all respects.

SCHOLIUM.—It appears, then, that the triangles can only differ when the equal angles B and E are acute, and the pair of sides opposite them less than the other pair of equal sides. In this case a relation holds important enough to form a separate proposition —of which we shall presently have occasion to make use.

PROP. II.—*If two triangles have two sides of one equal to two sides of the other, each to each, and the angles opposite one pair of equal sides equal; then, if*

the angles opposite the remaining pair of equal sides be unequal, their sum is equal to two right angles.

In the triangles A B C, D E F, let A B, A C, be equal to D E, D F, each to each; the angle B equal to the angle E, but the angle C greater than the angle F—so that by Prop. 1 C is obtuse and F acute. Then (Euc. I., 32) the angle E D F is greater than A. Make the angle E D G equal to A; then the triangle E D G is equal to the triangle A B C in all respects (Euc. I., 26). Hence D G is equal to A C, and there-

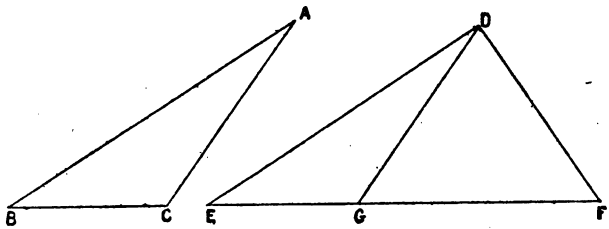


FIG. 45.

fore to D F; also, the angle D G E is equal to the angle C. Now since D G is equal to D F, the angle D G F is equal to the angle D F G. But D G F and D G E together make up two right angles; hence their respective equals F and C together make up two right angles.

COR. 1.—The triangle E D F exceeds the triangle A B C by the isosceles triangle D G F.

There are other elements, such as the area, altitude, and so on, which determine triangles. We shall have occasion, as we proceed, to notice how

triangles may be constructed when one or more such elements, combined perhaps with one or more of the six elements just considered, are given. But we may consider the relations discussed in Euc. I., 4, 8, 26, and in the above propositions, as the fundamental problems in the determination of triangles.

We proceed, therefore, to discuss the construction of triangles when certain of the six elements above considered are given.

Prop. 22 is the only one in which Euclid shows how to construct a triangle from given elements. But

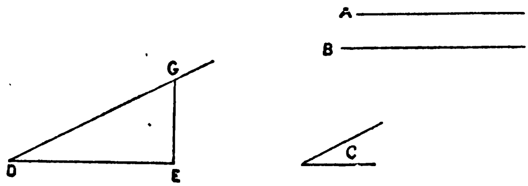


FIG. 46.

by means of Prop. 23 the following other cases can be solved.

PROP. III.—PROB. *To make a triangle having two sides equal to two given straight lines and enclosing a given angle.*

Let A, B, (Fig. 46) be the lines, C the given angle.

Take a straight line, D E, equal to A, and at the point D make the angle E D G equal to C. Take D G equal to B, and join G E. Then it needs no demonstration to show that D G E is the required triangle.

PROP. IV.—PROB. *To make a triangle having one side equal to a given straight line, and the angles adjacent to this side equal to two given angles whose sum is less than two right angles.*

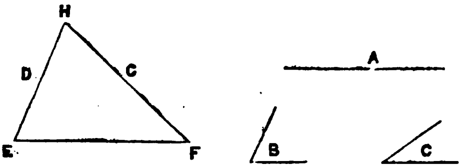


FIG. 47.

Let A (Fig. 47) be the given line, B, C the given angles. Take E F equal to A, at E make the angle F E D equal to the angle B, and at F make the angle E F G equal to the angle C. Then the sum of the angles F E D and E F G, being equal to the sum of the angles B and C, is less than two right angles. Hence E D and F G meet if produced far enough. Let them meet in H, then E H F is obviously the required triangle.

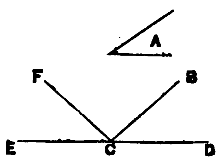


FIG. 48.

PROP. V.—PROB. *To construct a triangle having one side equal to a given straight line, and two angles not both adjacent to this side equal to two given angles whose sum is less than two right angles.*

If A and B C D (Fig. 48) are the given angles, we obtain the third angle of the triangle, and thus reduce the problem to Prop. 4., by producing D C to E, and making the angle E C F equal to the angle A. For the three angles of the triangle are together equal to the three angles E C F, F C B, and B C D together (Euc. I., 32). Therefore the third angle of the triangle is equal to F C B.

Another case remains, before proceeding to which, however, it will be well to establish the following theorem :—

PROP. VI.—*The perpendicular is the shortest line which can be drawn from a given point to a given*

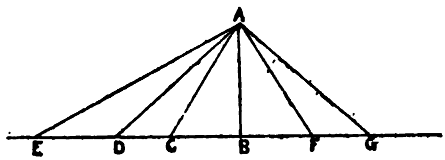


FIG. 49.

line ; and of others that which is nearer to the perpendicular is less than one more remote ; and not more than two equal lines can be drawn from the given point to the given straight line one on each side of the perpendicular.

Let A B (Fig. 49) be a straight line drawn from A perpendicular to E F ; and let A C, A D, A E be other straight lines from A to E F, in their order of distance from A B. Then, in the triangle A B C the angle A B C is a right angle ; hence the angle A C B is less

than a right angle (Euc. I., 17), and A B is therefore less than A C (Euc. I., 19). Also in the triangle A C D, A C D is an obtuse angle, being the supplement of the acute angle A C B; hence A D C is an acute angle, and A C is less than A D (Euc. I., 19). Similarly A D is less than A E, and so on.

Also, if A F be drawn so as to make the angle B A F equal to the angle B A C (Euc. I., 23), and meeting E G in F (Euc. I., 17, and Axiom 12), the triangles B A C and B A F are equal in every respect (Euc. I., 26), and therefore A F is equal to A C. Also no other line, as A G, can be equal to A C, that is to A F, because, then, two lines unequally remote from A B would be equal, which has been shown to be impossible.

PROP. VII.—PROB. *To construct a triangle having two sides equal to two given straight lines, and an angle opposite one of these sides equal to a given angle.*

Let A and B (Fig. 50) be the given lines, C the given angle; and let the side equal to B in the required triangle be that which is to be opposite to the angle equal to C.

Draw a line E F terminated towards E, and from E draw E D making the angle D E G equal to C. Take D E equal to A. From D draw D L perpendicular to E F. Then since D L is the shortest line connecting D with a point in E F (Prop. 6), if B be less than D L it is impossible to construct a triangle with the given elements. But if B be not less than D L, with centre D and radius equal to B, describe the circle

G H K cutting E F, produced if necessary towards E, in H and G.

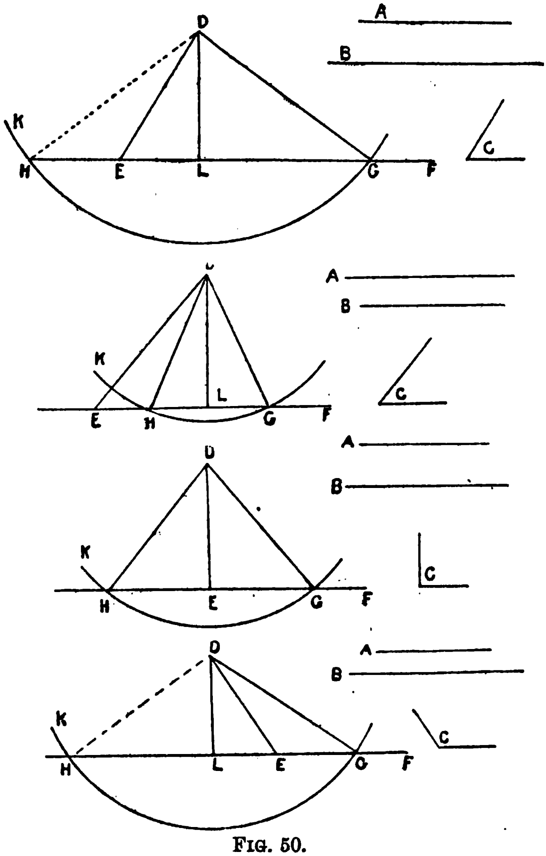


FIG. 50.

First suppose C an acute angle. Then if B greater than A, or D G, D H greater than D E, tl

points H and G lie on opposite sides of E (Prop. 6) Hence by joining D G we obtain the triangle D E G (and only this triangle), which has the required elements. If B is less than A, the points H and G lie on the same side of E (Prop. 6), and by joining D H and D G we obtain two triangles, D E H and D E G, each of which has the required elements. No demonstration is required in either case. If B be equal to D L, D G and D H coincide and there is but one solution.

If C be a right angle, the two triangles D E H and D E G (which are equal in all respects, yet cannot be superposed without turning one over) have the required elements.

Lastly, if C be obtuse, it is clear there is but one solution, since D G must be greater than D E that the circle K H G may meet E F in a point on E F. The triangle D E G clearly has the required elements.

In Euc. I., 5 and 6, we learn two properties of isosceles triangles. And it is to be noticed that the 9th, 10th, 11th, and 12th propositions involve, more or less, the properties of such triangles. It is clear, for instance, that the proof of Prop. 10 does not require that the triangle A C B should be equilateral, but only that A C, C B should be equal. So also, if D F, F E are equal in Prop. 11, the proof is sufficient. In Prop. 12, F C G is an isosceles triangle.

Now if we collect the properties of isosceles triangles involved in the three last-named propositions,

we see that they present themselves as shown in the following propositions.

PROP. VIII.—*The bisector of the vertical angle of an isosceles triangle bisects the base also.* This is established in the proof of Euc. I., 10.

PROP. IX.—*The bisector of the vertical angle of an isosceles triangle is at right angles to the base.* This is established in the proofs of Euc. I., 10 and 11.

PROP. X.—*The line joining the vertex of an isosceles triangle to the bisection of the base is at right angles to the base.* This is established in the proofs of Euc. I., 11 or 12.

It is obvious that the direct converse of each of these three propositions is also true.

But there are three indirect converse theorems which are often useful. They may be called the three fundamental tests of an isosceles triangle:—

PROP. XI.—*If the bisector of the vertical angle of a triangle also bisects the base, the other two sides are equal.*

In the triangle B A C (Fig. 51), let A D, bisecting the angle B A C, divide B C into two equal parts in D; then shall A B be equal to A C. In the triangles B A D, C A D, we have the sides B D, D A, equal to the sides C D, D A, each to each; and the angles B A D, C A D, which are opposite the equal sides, B D, C D, are likewise equal. Hence by Props. 1 and 2 the triangles are either equal in all respects, or else the angles B and C together make up two right angles. But the angles B and C, being two angles of a

triangle, are together less than two right angles. Hence the triangles A B D, A C D are equal in all respects. Therefore A B is equal to A C.

PROP. XII.—*If the line drawn from the vertex of a triangle to the bisection of the base is perpendicular to the base, the other two sides are equal.*

If (same figure) B D is equal to D C, and A D perpendicular to B C, the triangles A B D, A C D are equal in all respects by Euc. I., 4.

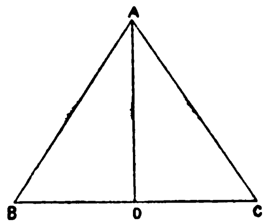


FIG. 51.

PROP. XIII.—*If the bisector of the vertical angle of a triangle is perpendicular to the base, the sides are equal.*

If (same figure) the angle B A D is equal to the angle D A C, and A D also at right angles to B C, the triangles A B D, A C D are equal in all respects by Euc. I., 26.

In Props. 13–15 Euclid exhibits properties of straight lines which are often useful in determining whether three or more points lie in a straight line, and also whether three or more lines pass through one point.

Props. 16–21 are of continual use in solving problems, as we have seen in Section I. and shall see farther on.

The following proposition is often useful:—

PROP. XIV.—*The difference between any two sides of a triangle is less than the third side.*

From A C, a side of the triangle A B C, cut off D C equal to B C; then the remainder A D is less than A B. For if A D be equal to (or greater than) A B, add D C to A D, and add B C, which is equal to D C, to A B; then A C is equal to (or greater than) A B and B C together, which is impossible (Euc. I.,

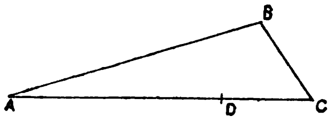


FIG. 52.

21). And in like manner the difference between A B, B C may be shown to be less than A C; and the difference between A C and A B less than B C.

It is well to note that in place of the general theorem which forms the latter part of Prop. 21 we may substitute the following useful proposition:—

PROP. XV.—*If from the extremities B C of the base B C of a triangle B A C the lines B D, C D be drawn to a point D within the triangle, then the angle B D C exceeds the angle B A C by the sum of the angles A B D and A C D.*

For the angle B D C is equal to the two angles

D C E, D E C (Euc. I., 32); that is, to the three angles D C E, A B E, and B A E (Euc. I., 32).

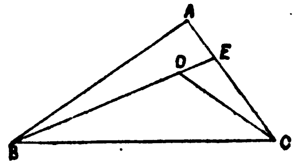


FIG. 53.

The following proposition is as often applicable as Prop. 23 : —

PROP. XVI.—PROB. *From a given point without a given line to draw a line which shall make with the given line an angle equal to a given rectilinear angle.*

Let A be the given point, B C the given line, and D the given angle.

From any point E in B C draw E F so that the angle F E C may be equal to the angle D (Euc. I., 23).

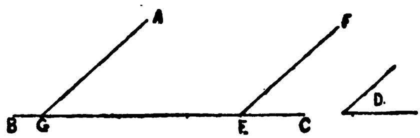


FIG. 54.

Through A draw A G parallel to F E. Then the angle A G E is equal to the angle F E C (Euc. I., 22), that is, to the angle D.

Props. 27–31 exhibit the properties of parallels. To these the following very useful property may be added :—

PROP. XVII.—*If there be any number of parallel lines A F, B G, C H, &c., and if any straight line A E meeting the parallels in the points A, B, C, D, &c., be divided into equal parts, A B, B C, C D, &c., then any other straight line F L, meeting the parallels in the points F, G, H, K, &c., will be divided into equal parts F G, G H, H K, &c.*

Draw A M and B N parallel to F L. Then the angle C B N is equal to the angle B A M (Euc. I., 30 and 29), and the angle A B M is equal to the angle

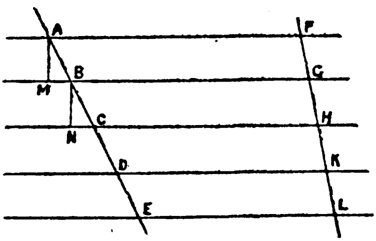


FIG. 55.

B C N (Euc. I., 29), also A B is equal to B C. Therefore the triangles A B M, B C N are equal in all respects (Euc. I., 26). Hence A M is equal to B N. But A M is equal to F G, and B N to G H (Euc. I., 34); therefore F G is equal to G H. And similarly it may be shown that G H is equal to H K, H K to K L, and so on. Hence F G, G H, H K, &c., are all equal.

Prop. 32 is very important, as are its corollaries. It is well to notice that the second corollary gives the easily remembered result that—

Each of the exterior angles of a regular polygon of n *sides is equal to* $\frac{4}{n}$*ths of a right angle.*

We may refer to this result as Euc. I., 32, Cor. 2, *Schol.* As an instance of its application, take the following:—

Each exterior angle of a regular heptagon is $\frac{4}{7}$ths of a right angle; hence each of the interior angles is equal to $\frac{10}{7}$ths (Euc. I., 13). Hence, also, the sum of the interior angles of a heptagon is equal to ten right angles.

The method here followed is the most convenient in cases of this sort, being so easily remembered.

To the properties established in Prop. 34, and their converse theorems, we may add the following:—

PROP. XVIII.—*The diagonals of a parallelogram bisect each other.*

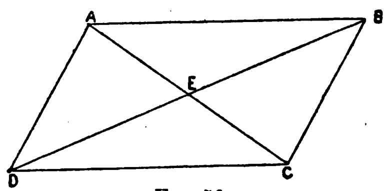


FIG. 56.

Let A C, B D, the diagonals of the parallelogram A B C D, intersect in E. Then shall A E be equal to E C, and D E to E B. In the triangles A E D, B E C, the vertical angles A E D, B E C are equal, the angle A D E is equal to the alternate angle

E B C, also A D is equal to B C. Hence (Euc. I., 26) the triangles are equal in all respects. Therefore A E is equal to E C, and D E to E B.

The converse of this property is also true. It is—*If the diagonals of a quadrilateral figure bisect each other the figure is a parallelogram.* It is clear that if A E is equal to E C, and D E to E B, then since the angle A E D is equal to the angle B E C (Euc. I., 15), the triangles A E D and B E C are equal in all respects (Euc. I., 4). Hence A D is equal to B C and the angle D A E is equal to the angle E C B; therefore A D is parallel to B C (Euc. I., 27); and since A D is equal and parallel to B C, A B is equal and parallel to D C (Euc. I., 33): therefore A B C D is a parallelogram.

We may refer to this proposition as Prop. 18 *Conv.*

Prop. 18 and its converse are propositions of great utility, and the student should always be on the watch to apply them in problems of a certain class. Examples will be given farther on.

Props. 35–41, and the converse theorems already discussed, are of frequent application.

Props. 42, 44, 45 have a useful application in surveying; for, by taking the given angle as a right angle, we learn how to reduce any rectilinear figure into a rectangle of equal area—and, if necessary, of given length or breadth. The fourteenth proposition of the Second Book shows us, further, how to construct a square equal to any given

rectilinear figure. Hence these propositions may be looked on as completing the theory of the quadrature of rectilinear figures. But the following method of reducing a rectilinear figure to a triangle of equal area is worth noticing for several reasons.

PROP. XIX.—PROB. *To reduce a rectilinear figure ABCDEF* (Fig. 57) *to a triangle of equal area, having its base in the line A F produced, and its vertex at D.*

Produce A F indefinitely either way to G and H. Join C A, through B draw B K parallel to C A, and

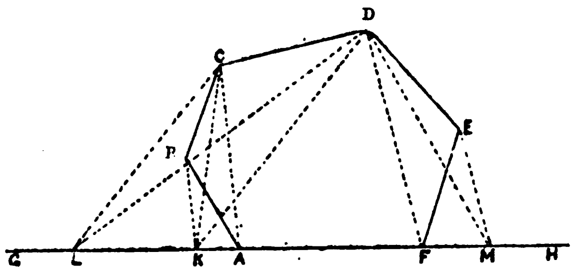


FIG. 57.

join C K; then the triangle A K C is equal to the triangle A B C, and (adding A F E D C to these equals) the figure F K C D E to the given figure A B C D E F. Again, join D K, draw C L parallel to D K and join D L: then the triangle K L D is equal to the triangle K C D, and (adding K F E D to these equals) the figure F L D E to the figure F K C D E, — that is, to the given figure. Lastly, join D F, draw E M parallel to D F and join D N; then the triangle D F M is equal to the triangle D E F, and the triangle

L D M to the quadrilateral L D E F—that is, to the given figure A B C D E F. And in like manner a figure of any number of sides may be reduced to a triangle.

It is hardly necessary to point out the importance of Props. 47 and 48. The following problem, forming a well-known 'puzzle,' exhibits an interesting proof of the 47th proposition:—

Let there be two squares, A B C D and E F G C, placed so that the sides D C, C G are contiguous and

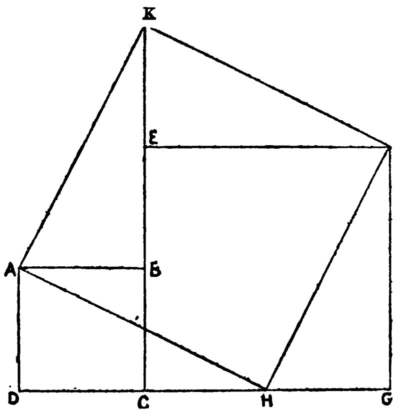


FIG. 58.

in one straight line, and therefore B C and E C coincident. It is required to draw two straight lines dividing the figure A D G F E B into three portions, which can be so combined as to form a single square.

Take D H equal to E F, so that H G is equal to

D C. Join D H and H F; then the figure is divided as required. For let the triangle A D H be placed as at A B K with its right angle coincident with the right angle A B E, the side A D being so placed as to coincide with A B; join K F; then B K being equal to D H, that is to E C, E K is equal to B C (that is, to H G); also E F is equal to F G, and the right angle K E F to the right angle H G F. Hence the triangle K E F is equal in all respects to the triangle H G F. Thus the figure A K F H is made up of three figures equal to those into which the figure A D G F E is divided by the lines A H, H F. Also A K F H is a square. For, the four triangles A D H, F G H, K E F, and A B K being obviously equal in all respects, the four lines A H, H F, F K, and K A are equal. And each of the angles of A K F H is a right angle. For, the angle K F H is equal to E F G since K F E is equal to H F G; K A H is equal to D A B since K A B is equal to D A H; and A K F is the sum of the angles A K B, E K F—that is, of the angles K F E, E K F, which together make up a right angle (Euc. I., 32): hence, also, A H F is a right angle, and A H F K is a square.

Of course this problem is, in effect, a proof of Prop. 47.

We shall now proceed to some problems on the subject of the First Book, which are of great utility and importance.

Prop. XX.—*If the three sides A B, B C, C A of the triangle A B C be bisected in the points D, E, and*

F, the three lines D E, E F, and F D are respectively parallel to the sides C A, A B, and B C of the triangle A B C, and equal to the halves of these lines, respectively.

For, produce F D to G, making G D equal to D F, and join B G, A G. Then by Prop. 18 *Conv.*, since A B, G F are bisected in D, A G B F is a parallelogram. Therefore B G is equal and parallel to A F—that is, to F C. Therefore (Euc. I., 33) G F is equal and parallel to B C. But G F is double of

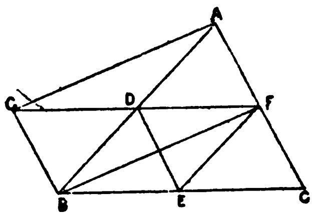


FIG. 59.

D F (*const.*) and B C of E C (*hyp.*); therefore D F is equal to E C or B E. And in like manner it may be shown that D E is parallel to A C and equal to A F or F C; and that E F is parallel to A B and equal to A D or D B.

COR.—The four triangles A D F, E F D, B D E, and F E C are equal in all respects (Euc. I., 8 and 4), and equiangular to the triangle A B C.

PROP. XXI.—*Let A C B be a right-angled triangle, C being the right angle, and let A B be bisected in D; then shall A D, D C, and D B be all equal.*

Bisect A C in E and join D E; then D E is parallel to B C (Prop. 20). Therefore the angle A E D is equal to the interior A C B (Euc. I., 2);

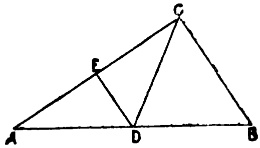


FIG. 60.

that is, A E D is a right angle. Thus the line D E is at right angles to and also bisects the side A C of the triangle A D C; therefore (Prop. 13) A D is equal to D C. Similarly D B is equal to D C.

PROP. XXII.—*If the two equal and parallel straight lines A B, D C be bisected in the points E and F, then C E and A F trisect the line D B in the points G and H.*

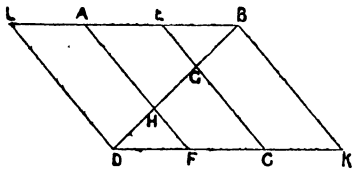


FIG. 61.

Produce B A to L, making A L equal to A E or E B; and produce D C to K, making C K equal to D F or F C. Join L D and B K. Then since the lines L A, A E, and E B are equal to each other and also to the three lines D F, F C, and C K; therefore

(Euc. I., 33) the lines L D, A F, E C, and B K are parallels. But L B is trisected in A and E where it meets the parallels. Therefore B D is trisected in G and H (Prop. 17).

COR.—If A B, C D were divided into any number, n, of equal parts, and lines drawn from C to the division-point nearest B, from the division-point nearest C to the second division-point from B, from the second division-point from C to the third division-point from B, and so on, these lines would divide the diagonal B D into $(n + 1)$ equal parts.

The proof of the corollary would be similar to that of the proposition, C K and A L being taken equal to any one of the n equal parts of B A and C D.

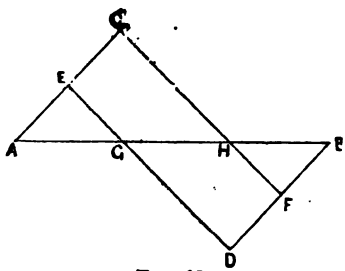


FIG. 62.

PROP. XXIII.—*To trisect a given straight line.*

Let A B be the given straight line. From A and B draw, in opposite directions, the equal parallels A C and B D. Bisect A C in E and B D in F. Join D E and C F; intersecting A B in G and H. Then A B is trisected in G and H (Prop. 22).

Cor.—In like manner we can divide a straight line into any number, n, of equal parts. For we have only to draw two unlimited parallels in opposite directions from the points A and B, and to take off, from A and B $(n - 1)$ equal divisions (of any length) along these lines. Then, joining the points of division in the manner indicated in Prop. 22, Cor., A B will be divided into n equal parts.

Another method of trisecting a line is usually given. This will be presented as a problem farther on.

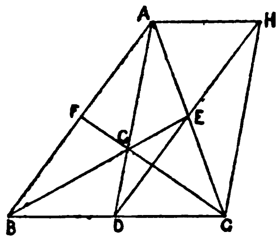


Fig. 63.

Prop. XXIV.—*If the three sides B C, C A, and A B of the triangle A B C be bisected in the points D, E, and F; the three lines A D, B E, and C F pass through one point, which is a point of trisection of each of the three lines.*

Let B E intersect A D in G; join D E and produce to H, making E H equal to E D. Join A H, H C. Then since A C and D H bisect each other, A H C D is a parallelogram (Prop. 18 *Conv.*). Therefore A H is equal and parallel to D C—that is,

to B D (*hyp.*). Hence H D is equal and parallel to A B (Euc. I., 33), and therefore, since D E is equal to E H, D G is a third part of A D (Prop. 22.). Similarly it may be shown that C F cuts off from A D a third part, towards D—that is, C F passes through the point G. And as G D has been shown to be a third part of A D, so may G E be shown to be a third part of B E, and F G to be a third part of F C.

PROP. XXV.—*If the triangles A B C, D E F be on the equal bases B C, E F, and between the same parallels,*

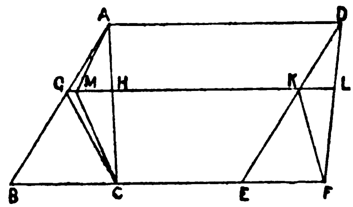


FIG. 64.

A D and B F, and G H K L be drawn parallel to B F, meeting A B, A C, D E, and D F, in the points G, H, K, and L respectively, G H shall be equal to K L.

For, if not, one of these lines must be greater than the other. Let G H be the greater, and from H G cut off H M equal to K L. Join G C, K F, A M, and M C.

Then, since M H is equal to K L, the triangles A M H and D K L are equal (Euc. I., 38), and so are the triangles M H C and K F L. Therefore the

triangle A M C is equal to the triangle D K F. But the triangle B G C is equal to the triangle K E F (Euc. I., 38). Therefore the triangles G B C, A M C are together equal to the triangle D E F—that is, to the triangle A B C. But this is absurd. Therefore G H and K L are not unequal. Therefore they are equal.

PROP. XXVI.—*Let the sides A B, A C of the triangle A B C be bisected in the points F and E, and let F P and E P, at right angles to A B, A C, meet in P: then the line drawn from P at right angles to B C shall*

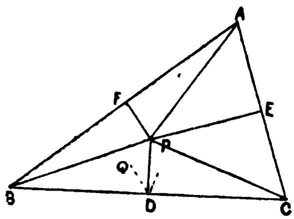


FIG. 65.

bisect B C in D; and the line drawn from P to bisect B C shall be at right angles to B C.

For, because A F is equal to F B and F P is common to the two triangles A F P, B F P and at right angles to A B, these triangles are equal in all respects. Therefore B P is equal to A P. In like manner A P is equal to P C. Therefore B P is equal to P C. Hence, in the isosceles triangle B P C, P D at right angles to B C bisects B C in D; and *vice versâ.*

PROP. XXVII.—*The three lines bisecting the sides of a triangle at right angles pass through one point.*

If F P, E P, two of these bisectors (same fig.), meet in P, the third bisector, through D, shall pass through P. For if it has any other position as D Q, the angle B D Q is a right angle. But by Prop. 26, P D is at right angles to B C. Therefore the angle P D B is equal to the angle Q D B; which is absurd. Therefore the perpendicular from D passes through the point P.

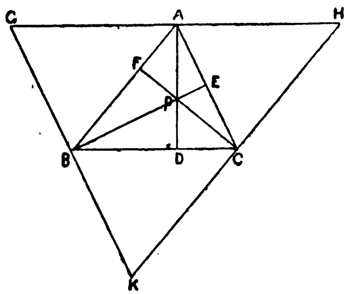


FIG. 66.

PROP. XXVIII.—*The lines drawn from the angles of a triangle at right angles to the opposite sides pass through one point.*

Let A D, B E, C F (Fig. 66) be perpendiculars on B C, C A, A B, the sides of the triangle A B C. Then shall A D, B E, C F pass through one point.

Through A draw G H parallel to B C; through B draw G B K parallel to A C; and through C draw K C H parallel to A B. Then G C is a parallelogram

and therefore G A is equal to B C. Similarly A H is equal to B C. Therefore G A is equal to A H. In like manner K C is equal to C H, and G B to B K. But since the angle G A D is equal to the alternate angle A D C, D A is at right angles to G H; similarly B E is at right angles to G K; and C F is at right angles to K H. Hence, by Prop. 27, A D, B E, and C F pass through one point, P.

COR.—If through the angles of a triangle lines are drawn parallel to the opposite sides, the sides of the triangle thus formed are bisected at the angles of the first triangle, and form a triangle four times as great as the first triangle (Prop. 20).

On account of the importance of Proposition 28 we shall give other proofs of it presently.

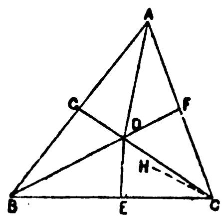


FIG. 67.

PROP. XXIX.—*In the triangle A B C* (Fig. 67), *let the lines A D, B D bisect the angles B A C, A B C respectively; then shall C D bisect the angle A C B.*

Draw D E, D F, and D G perpendicular to B C, C A, and A B respectively. Then the triangles

A G D, A F D are equal in all respects (Euc. I., 28); therefore D G is equal to D F. Similarly D E is equal to D G. Hence D E is equal to D F: and in the triangles D C E, D C F, D E is equal to D F; the angles D E C, D F C, opposite to the common side D C, are equal, being right angles; and the angles D C E, D C F, opposite to the equal sides D E and D F, are both acute (Euc. I., 17), since the angles at F and E are right angles. Hence the triangles D C E, D C F are equal in all respects; therefore the angle D C B is equal to the angle D C A.

PROP. XXX.—*The three lines bisecting the three angles of a triangle pass through one point.*

If A D, B D (same fig.), two of the bisectors, meet in D, the third must pass through D; for if it had any other position, as C H, then the angle E C H would be equal to half the angle B C A. But, by the preceding proposition, D C E is equal to half the angle B C A. Therefore the angle D C E is equal to the angle H C E; which is absurd. Therefore the three bisectors all pass through one point.

PROP. XXXI.—*If two sides A B, A C* (Fig. 68) *of the triangle A B C be produced to D and E, and the angles D B C, E C B be bisected by the lines B F, C G, these lines will meet, and the line joining the point in which they meet, with A, will bisect the angle B A C.*

For the angle D B C is less than two right angles, and therefore the angle F B C is less than one right angle. Similarly the angle G C B is less than a right angle. Therefore the two angles F B C, G C B

are together less than two right angles, and B F, C G will meet if produced far enough. Let them meet in H, and draw H K, H L, H M, perpendiculars on A D, B C, and A E. Then, the triangles H B K, H B L, are equal in all respects (Euc. I., 4); therefore, H K is equal to H L. Similarly it may be shown that H M is equal to H L. Hence H K is equal to H M. Therefore in the triangles H K A, H M A, K H is equal to H M, the angles A K H,

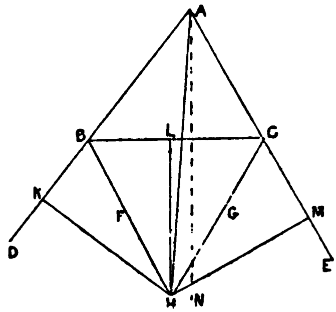


FIG. 68.

A M H, opposite to the common side H A, are equal (being right angles), and the angles K A H, M A H, opposite the equal sides K H, M H, are both acute—Euc. I., 17 (since the angles at K and M are right angles); hence the triangles K A H, M A H are equal in all respects; and therefore the angle K A H is equal to the angle M A H.

COR.—If two exterior angles of a triangle are bisected, the intersection of the bisecting lines is equidistant from the three sides of the triangle.

PROP. XXXII.—*If two sides of the triangle A B C (same figure) be produced to D and E, the lines bisecting the three angles D B C, B A C, and E C B, will all pass through one point.*

Let the bisectors of the angles D B C, and B C E meet in H; then the bisector of the angle B A C must pass through H. For if it had any other position as A N, the angle N A M would be equal to half the angle B A C. But, by the preceding proposition, the angle H A M is equal to half the angle B A C. Hence the angle H A M is equal to the angle N A M; which is absurd.

COR.—The bisectors of the angles B A C, B C E intersect, and the line joining their intersection with B bisects the angle D B C.

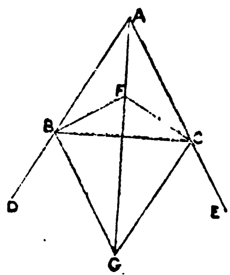


FIG. 69.

PROP. XXXIII.—*In the triangle A B C* (Fig. 69), *let B F, C F be the bisectors of the angles A B C, A C B; B G, C G the bisectors of the exterior angles D B C, B C E: then the points A, F, and G lie in a right line.*

For by Prop. 30 the point F lies on the bisector

of the angle B A C, and by Prop. 32 the point G lies on the same bisector. Hence the points A, F, and G lie in one straight line.

PROP. XXXIV.—*Let the sides A B, B C, C A of the triangle B A C* (Fig. 70) *be severally produced both ways, to D, E, F, G, H, and K, and let A L, B L, and C L be the bisectors of the angles B A C, C B A, and A C B : then the lines M A N, M B P, and P C N, drawn*

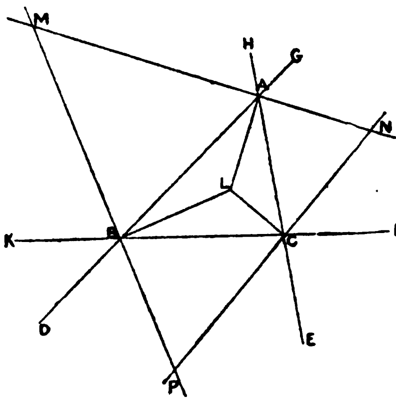


FIG. 70.

at right angles to A L, B L, and C L, shall bisect the angles H A B and G A C, A B K and D B C, B C E and A C F.

For the right angle L B M is equal to the right angle L B P, whereof the portions A B L, L B C are equal. Hence the remaining angle M B A is equal to the remaining angle P B C. But the angle M B A is equal to the vertical angle D B P, and C B P to

M B K (Euc. I., 15). Hence the four angles A B M, M B K, D B P, P B C are all equal; or P M bisects both the angles A B K and C B D. Similarly P N bisects both the angles A C F and B C E; and M N bisects both the angles C A G and B A H.

Cor. 1.—A line which bisects any angle bisects also (when produced) the vertical angle.

Cor. 2.—The bisectors of the two pairs of vertical angles formed by two intersecting lines are at right angles to each other.

Prop. XXXV.—*Let B C, a side of the triangle A B C, be produced either way to D and E, and let B A, C A be produced respectively to G and F: then*

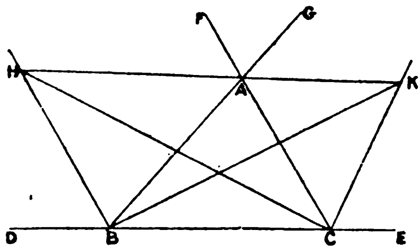


Fig. 71.

if B H, C H, the bisectors of the angles A B D, A C B, meet in H, and C K, B K, the bisectors of the angles A C E, A B C meet in K, the points H, A, and K lie in a straight line, and this line bisects the angles F A B and G A C.

For, by Prop. 32, Cor., A H bisects the angle F A B and A K bisects the angle G A C. But the angle F A B is equal to the angle G A C. Therefore

the angles H A B, G A K, being the halves of these equal angles, are equal. But H A B and G A K are vertical angles. Therefore H A and A K are in the same straight line (Euc. I., 15, *Conv.*), and it has been shown that H A K bisects the angles F A B and G A C.

The *method* of the following proof of Prop. 18 is worth noticing.

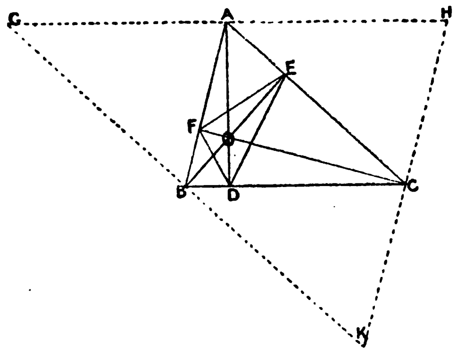


FIG. 72.

First, let the triangle A B C (Fig. 72) be acute-angled. Draw A D perpendicular to B C and B E perpendicular to A C: then if it can be shown that C O produced cuts A B at right angles it will be obvious that Prop. 28 is established, since there is only one perpendicular from C on A B.

Join D E. Then the angles of the quadrilateral O E C D are together equal to four right angles (Euc. I., 32, Cor. 1); therefore since the angles O E C,

O D C are right angles, the angles D O E and D C E are together equal to two right angles. But D C E is acute (*hyp.*), therefore D O E is obtuse; and the angles O E D, O D E are therefore together less than a right angle (Euc. I., 32). Hence if we make the angles O E F, O D F equal to O E D, O D E respectively, the two angles D E F, E D F are together less than two right angles (being double of the two angles O E D, O D E together). Hence E F and D F meet as shown in the figure. Join O F. Now in the triangle F E D, O E and O D are the bisectors of the angles F E D, F D E; therefore A E C and B D C are the bisectors of the angles external to F E D, F D E (Prop. 34.) Hence by Prop. 35, F lies on the line A B, and A F B is the bisector of the angles external to D F E. Also F O is the bisector of the angle E F D (Prop. 29). Therefore A B is at right angles to F O (Prop. 34, Cor. 2). But the points C, O, and F are in one straight line (Prop. 33). Hence C O F is at right angles to A B.

Secondly, take the case of the obtuse-angled triangle A O B. Produce A O to D and draw B D perpendicular to A D. Produce B O to E and draw A E perpendicular to B E. Then the angles A E D and B D E are each greater than a right angle, so that A E and B D must meet if produced towards E and D. Let them meet in C. Then we have to show that a perpendicular from O on A B will, if produced, pass through C. Now, since O E C and O D C are right angles and D O E is obtuse, the

angle E C D is acute (Euc. I., 32, Cor. 1); also, since A E B and A D B are right angles, the angles E A B and D B A are acute (Euc. I., 17). Hence in the acute-angled triangle A B C, the perpendiculars B E and A D intersect on the perpendicular from C on A B. That is, the perpendicular from C on A B passes through O; or, in other words, the perpendicular from O on A B, produced beyond O, passes through C, the point of intersection of A E and B D produced.

SCHOLIUM.—It is worthy of notice that if we take any triangle, G H K, bisect its sides in the points A, B, C, and form the triangle A B C, and again draw the perpendiculars A D, B E, and C F, and form the triangle D F E, then the three important properties contained in Props. 27, 28, and 30 are illustrated together; since the same three lines A D, C F, and B E are at once the bisectors of the angles of the triangle D F E, the perpendiculars from the angles on the opposite sides of the triangle A B C, and the rectangular bisectors of the sides of the triangle G H K.

PROP. XXXVI.—*The area of a trapezium is equal to half that of a parallelogram whose base is equal to the sum of the two parallel sides of the trapezium, and whose altitude is equal to the distance between them.*

Let A B C D (Fig. 73) be a trapezium, the sides A B C D being parallel. Produce A B to F, making A F equal to D C; and C D to E, making D E equal to A B; and join E F.

Then F B is equal to F A and A B together; that is, to D C and D E together (*const.*), that is, to E C; and F B is also parallel to E C. Hence E F is equal to B C (Euc. I., 33); and F C is therefore the parallelogram, having a base equal to the sum of the parallel sides A B, D C, and an altitude equal to the distance of A B from D C. Now, it is obvious that the trapezium A B C D is equal to half the parallelogram F C, since F E D A is a trapezium equal to A B C D in all respects.

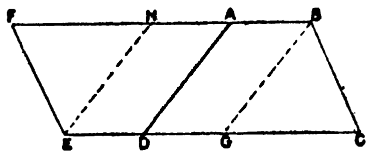


FIG. 73.

[The equality of F E D A and A B C D is so obvious as scarcely to need demonstration. It is clear that the lines B G, E H, drawn parallel to A D, divide F C into the pair of equal parallelograms H D, D B, and the pair of equal triangles E H F and B G C.]

PROP. XXXVII.—*The area of a quadrilateral is equal to half that of the circumscribing parallelogram whose sides are parallel to the diagonals of the quadrilaterals.*

Let the parallelogram E F G H (Fig. 74) circumscribing the quadrilateral A B C D have its sides E F, H G, parallel to A C, and F G, E H parallel to

B D; then shall the quadrilateral A B C D be equal to half the parallelogram E F G H.

Let A C, B D intersect at O. Then E O is a parallelogram; therefore the triangle A B O is equal to half of E O (Euc. I., 34); similarly the triangles B O C, C O D, D O A are equal to half O F, O G, and O H respectively. Hence the whole figure A B C D is equal to half the parallelogram E F G H.

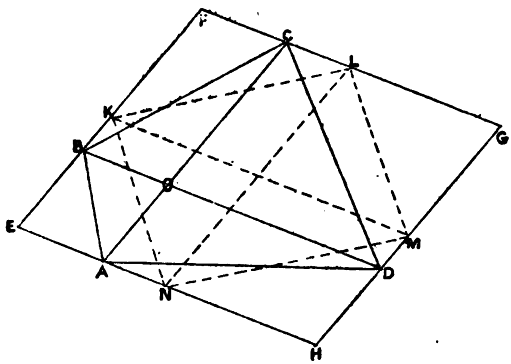


FIG. 74.

COR. 1.—The parallelogram E F G H is equal to twice any quadrilateral as A B C D, so inscribed that the diagonals A C and B D are parallel to E F and F G respectively.

COR. 2.—Let K, L, M, N be the bisections of E F, F G, G H, and H E. Then K L and N M are each parallel to E G by Prop. 20, and therefore to each other; and similarly L M is parallel to K N. Hence K L M N is a parallelogram; and the diagonals

K M, L N are inclined to each other at the same angle as those of the quadrilateral A B C D. Now, by Cor. 1, E F G H is double both of A B C D and K L M N; hence A B C D is equal to K L M N; that is, a quadrilateral is equal to the parallelogram having diameters equal to those of the quadrilateral and equally inclined to each other.

Cor. 3.—Quadrilaterals having equal diameters (each to each), equally inclined, are equal to each other.

Prop. XXXVIII.—*Parallelograms on equal bases and of equal altitude are equal to each other.*

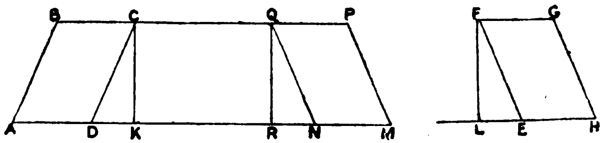


Fig. 75.

Let A B C D and E F G H (Fig. 75) be parallelograms on equal bases A D, E H, and of equal altitudes C K and F L. The parallelogram B D shall be equal to the parallelogram F H.

Produce A D to M, take N M equal to E H or A D, and complete the parallelogram Q M equal in all respects to F H. Draw Q R perpendicular to A M and join C Q.

Then the triangles Q N R and F E L are equal in all respects (Euc. I., 26). Hence Q R is equal to F L—that is, to C K. But since Q R and C K are

each of them perpendicular to A M, they are parallel to each other. Hence C Q is parallel to A M, and the parallelograms B D and Q M are equal (Euc. I., 36). But Q M is equal to F H. Therefore B D is equal to F H.

Prop. XXXIX.—*Triangles on equal bases and of the same altitude are equal to one another.*

The triangles are the halves of parallelograms on equal bases and of the same altitude; and are therefore equal by the preceding proposition.

Prop. XL.—*Equal parallelograms on equal bases are of the same altitude.*

Prop. XLI.—*Equal triangles on equal bases are of the same altitude.*

Prop. XLII.—*Equal parallelograms of equal altitude are on equal bases.*

Prop. XLIII.—*Equal triangles of equal altitude are on equal bases.*

These four propositions require no demonstration; for if we assumed that the altitudes in the two former, or the bases in the two latter, were unequal, an obvious absurdity would result.

Prop. XLIV.—*Let A B C D* (Fig. 76) *be a quadrilateral figure, and from B, D, the extremities of one diameter, let B E, D F be drawn, perpendicular to A C, the other diameter. The area of the quadrilateral A B C D shall be equal to the area of a right-angled triangle having one side equal to A C and the other equal to the sum of the lines B E and D F.*

Through A and C draw the lines G A H and

K C L at right angles to A C; and through B and D draw the lines G B K, H D L parallel to A C, and therefore at right angles to G H and K L.

Then the rectangle G C is equal to twice the triangle A B C, and the rectangle A L is equal to twice the triangle A D C. Therefore the whole

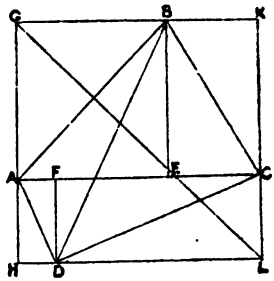


FIG. 76.

rectangle G K L H is equal to twice the quadrilateral A B C D. But the rectangle G H L K is equal to twice the triangle G H L. Therefore the quadrilateral A B C D is equal to the right-angled triangle G H L, which has one side, H L, equal to A C, and the other side, G H, equal to the sum of the lines B E and D F.

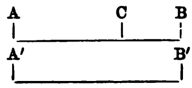
Notes on Euclid's Second Book.

The Second Book of Euclid affords a good illustration of what might be done in the way of simplifying Euclid, while retaining his arrangement of

propositions, and scarcely departing from his method. Thus this book might be presented as follows :—

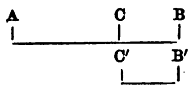
PROP. I.—Enunciation and proof as in Euclid—only, in a simplified Euclid, the *statement* which now heads the demonstration can be made the *enunciation*, as in the propositions which follow.

PROP. II.—*Let a straight line A B be divided into any two parts in the point C : then the rectangle A B, B C, together with the rectangle A B, A C, shall be equal to the square on A B.*



Let A′ B′ be equal to A B. Then by Prop. 1, Rect. A C, A′ B′ + rect. B C, A′ B′ = rect. A B, A′B′; that is, rect. A C, A B + rect. B C, A B = sq. on A B.

PROP. III.—*Let a straight line A B be divided into any two parts in the point C : then the rectangle A B, B C shall be equal to the rectangle A C, C B together with the square on B C.*



Let C′ B′ be equal to B C. Then by Prop. 1, Rect. A B, B′ C′ = rect. A C, C′ B′ + rect. B C, B′ C′; that is,

Rect. A B, B C = rect. A C, C B + sq. on B C.

PROP. IV.—*Let the straight line A B be divided into any two parts in C : then the square on A B shall*

be equal to the squares on A C, C B together with twice the rectangle A C, C B.

By Prop. 2,

Rect. A C, C B + sq. on A C = rect. A B, A C,

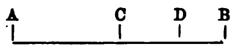
Rect. A C, C B + sq. on B C = rect. A B, B C,

$\therefore$ 2 Rect. A C, C B + sqs. on A C, B C = rect. A B, A C + rect. A B, B C = sq. on A B (by Prop. 2).

Cor.—If A B is bisected in C, the sq. on A B is equal to four times the sq. on A C.

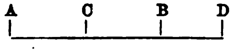
Prop. V.—*Let the straight line A B be divided into two equal parts in C, and into two unequal parts in D: then the rectangle A D, D B together with the square on C D shall be equal to the square on C B.*

By IV.,



Sq. on C B = sq. on C D + sq. on D B + rect. C D, D B + rect. C D, D B
= sq. on C D + rect. C B, D B + rect. C D, D B (by III.)
= sq. on C D + rect. A C, D B + rect. C D, D B ($\because$ C B = A C)
= sq. on C D + rect. A D, D B (by I.)

Prop. VI.—*Let the straight line A B be bisected in C and produced to D: then the rectangle A D, D B,*



together with the square on C B, shall be equal to the square on C D.

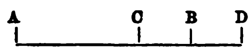
By IV.,

Sq. on C D = sq. on C B + sq. on B D + rect. C B, B D
+ rect. C B, B D
= sq. on C B + rect. C D, B D
+ rect. C B, B D
= sq. on C B + rect. A D, D B
(by I. $\because$ C B = A C).

PROP. VII.—*Let the straight line A B be divided into any two parts in C: then the squares on A B, B C are equal to twice the rectangle A B, B C together with the square on A C.*

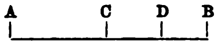
By IV., Sq. on A B = sq. on A C + sq. on B C
+ 2 rect. A C, C B
$\therefore$ sqs. on A B, B C = sq. on A C + 2 sq. on B C
+ 2 rect. A C, C B
= sq. on A C + 2 rect. A B, B C
(by III.).

PROP. VIII.—*Let the straight line A B be divided into any two parts in C, and produced to D, so that B D = B C: then shall the square on A D be equal to four times the rectangle A B, B C together with the square on A C.*



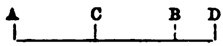
By IV., Sq. on A D = sq. on A C + sq. on D C
+ 2 rect. A C, C D
= sq. on A C + 4 sq. on C B
+ 4 rect. A C, C B (by Cor. to IV., and I.)
= sq. on A C + 4 rect. A B, B C
(by III.).

PROP. IX.—*Let the straight line A B be divided into two equal parts in C, and into two unequal parts in D: then the squares on A D, D B are together double the squares on A C, C D.*



By IV., Sq. on A D = sq. on A C + sq. on C D
+ 2 rect. A C, C D;
$\therefore$ sqs. on A D, B D = sq. on A C + sq. on C D
+ 2 rect. C B, C D
+ sq. on D B
= sq. on A C + sq. on C D
+ sq. on C B
+ sq. on C D (by VII.)
= 2 sq. on A C + 2 sq. on C D
($\because$ C B = A C)

PROP. X.—*Let the straight line A B be bisected in C, and produced to D: then the squares on A D, D B shall be double of the squares on A C, C D.*



By IV., Sq. on A D = sq. on A C + sq. on C D
+ 2 rect. A C, C D;
$\therefore$ sqs. on A D, B D = sq. on A C + sq. on C D
+ 2 rect. C B, C D
+ sq. on B D
= sq. on A C + sq. on C D
+ sq. on C B + sq. on C D
(by VII.)
= 2 sq. on A C + 2 sq. on C D
($\because$ C B = A C).

PROP. XI.—PROB. *Let A B be any straight line; it is required to divide A B into two parts, so that the rectangle by the whole and one part shall be equal to the square on the other.*

Bisect A B in C, and from A draw A D perpendicular to A B, and = A C. Join B D. With D as centre describe the circular arc A E, cutting D B in E, and with B as centre describe the circular arc E F, cutting B A in F. Then shall the rectangle A B, A F, be equal to the square on B F.

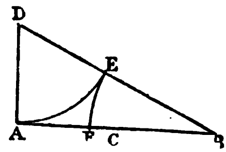


FIG. 77.

For sq. on D A + sq. on A B = sq. on D B
= sq. on D E + sq. on E B
+ 2 rect. D E, E B;

$\therefore$ sq. on A B = sq. on F B + 2 rect. A C, F B
($\because$ D E = A D = A C, and B E = B F).

That is, rect. A B, A F + rect. A B, F B
= sq. on F B + rect. A B, F B;

$\therefore$ rect. A B, A F = sq. on F B.

PROP. XII.—*Let A B C be an obtuse-angled triangle, A C B being the obtuse angle, and from A let A D be drawn perpendicular to B C produced; then the square on A B is equal to the squares on B C, C A together with twice the rectangles B C, C D.*

Sq. on A B = sq. on A D + sq. on B D
= sq. on A D + sq. on C D
+ sq. on B C + 2 rect. B C, C D
= sq. on A C + sq. on B C
+ 2 rect. B C, C D

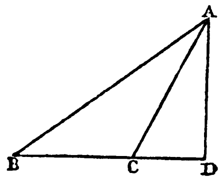


FIG. 78.

PROP. XIII.—*Let A B C be a triangle having the angle B acute, and draw A D perpendicular to B C, one of the sides containing the acute angle : then the squares on A B, B C are together equal to the square on A C with twice the rectangle B D, B C.*

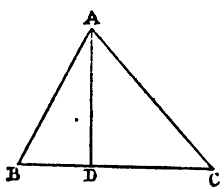


FIG. 79.

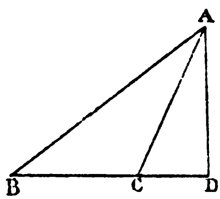


FIG. 80.

First, let D fall between B and C (Fig. 79.).
Then, sq. on A B = sq. on A D + sq. on B D
and sq. on B C = sq. on D C + sq. on B D
+ 2 rect. B D, D C ;

$\therefore$ sq. on AB + sq. on BC = sq. on AD + sq. on DC
+ 2 sq. on BD + 2 rect. BC, DC
= sq. on AC + 2 rect. BD, BC.

Next, let D fall on BC produced (Fig. 80). Then
sq. on AB = sq. on AD + sq. on BD
= sq. on AD + sq. on DC + sq. on BC
+ 2 rect. BC, CD;
$\therefore$ sq. on AB + sq. on BC = sq. on AC
+ 2 sq. on BC + 2 rect. BC, CD.
= sq. on AC + 2 rect. BC, BD.

Last, the case in which D coincides with C needs no demonstration, and has no interest, being to all intents identical with Prop. 47, Book I.

PROP. XIV.—*To describe a square that shall be equal to a given rectilinear figure.*

This proposition cannot be more briefly dealt with, at this place, than as Euclid treats it. But it is not really wanted till after properties have been established in Book III. by which the demonstration may be shortened.

Let us now, however, make a more careful analysis of this book:—

In the Second Book, Euclid deals with the relations between the rectangles contained by straight lines and the parts into which they may be divided. The method he adopts is somewhat cumbrous—so far, at least, as Problems 2–10 are concerned. The student must not deal with problems on the Second Book in Euclid's manner. In order to illustrate the proper method of dealing with such deductions we give

new solutions of Propositions 4 to 10, premising that Propositions 2 and 3 are particular cases of Proposition 1.

Euc. II., Prop. 4 should be thus established :—

Let A B be the given straight line divided into any two parts in the point C : the square on A B shall be equal to the squares on A C, C B together with twice the rectangle A C, C B.

By Prop. 2 the square on A B is equal to the rectangle A B, A C together with the rectangle A B, B C.

But by Prop. 3 the rectangle A B, A C is equal to the rectangle A C, C B together with the square on A C; and the rectangle A B, B C is equal to the rectangle A C, C B together with the square on B C.

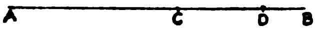
Hence the square on A B is equal to twice the rectangle A C, C B together with the squares on A C and C B.

Cor.—If a straight line be divided into two equal parts the square on the whole line is equal to four times the square on either half.

Prop. 4 may be enunciated thus:—*If a straight line be divided into any two parts, the square on one part is less than the square on the whole line by twice the rectangle contained by the parts together with the square on the other part.*

Prop. 5 should be established thus :—Let the straight line A B be divided into two equal parts in C,

and into two unequal parts in D; then the rectangle A D, D B, together with the square on C D, shall be equal to the square on C B.



Since A D is made up of A C, C D, whereof A C is equal to C B, the rectangle A D, D B is equal to the rectangle C D, D B together with the rectangle C B, D B (Euc. II., 1)—that is, to twice the rectangle C D, D B together with the square on D B (Prop. 3). Add the square on C D. Then the rectangle A D, D B together with the square on C D is equal to twice the rectangle C D, D B together with the squares on C D, D B—that is, to the square on C B (Prop. 4).

Cor.—Since the square on A C or C B is equal to the rectangle A C, C B, it follows that if a straight line is divided into unequal parts the rectangle contained by these is less than the rectangle contained by the halves of the line; and also (the deficiency being the square on C D) that the more unequal the parts the smaller is the rectangle contained by them.

Prop. 6 should be established thus:—

Let the straight line A B be bisected in C, and

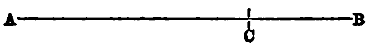
produced to D; then the rectangle A D, D B together with the square on C B shall be equal to the square on C D.

Produce D A towards A to E, making E A equal to B D, so that C E is equal to C D and B E to A D.

Then by Prop. 5, the rectangle E B, D B together with the square on C B is equal to the square on C D; that is, the rectangle A D, D B together with the square on C B is equal to the square on C D.

Props. 5 and 6 may be included in one enunciation thus:—*The rectangle contained by the sum and difference of two straight lines is equal to the difference of their squares (A C, C D being the two lines referred to)*; or thus:—*Taking A D and B D as the two lines of reference, the rectangle contained by two lines is equal to the square of half their sum diminished by the square of half their difference.*

Prop. 7 is proved thus:—Let the straight line A B be divided into any two parts in C; the squares on



A B, B C shall be equal to twice the rectangle A B, B C together with the square on A C.

The square on A B is equal to the squares on A C, C B, with twice the rectangle A C, C B (Prop. 4). Add the square on C B. Then the squares on A B, C B are together equal to the square on A C, twice the square on C B, and twice the rectangle A C, C B; that is, to the square on A C together with twice the rectangle A B, B C (Prop. 3).

Prop. 7 may be enunciated thus:—*The square on the difference of two lines (A B and B C) is less than the*

sum of the squares on those lines by twice the rectangle contained by them.

Prop. 8 is proved thus :—Let the straight line A B be divided into any two parts in C; then four

times the rectangle A B, B C, together with the square on A C, is equal to the square on the straight line made up of A B and B C together.

Produce A B to D, making B D equal to B C, so that A D is the line made up of A B and B C together. Then the square on A D is equal to the squares on A C, C D together with twice the rectangle A C, C D. But the square on C D is equal to four times the square on C B (Prop. 4, Cor.), and the rectangle A C, C D is equal to twice the rectangle A C, C B (Prop. 1). Hence the square on A D is equal to the square on A C together with four times the square on C B and four times the rectangle A C, C B; that is, to the square on A C and four times the rectangle A B, C B (Prop. 3).

Cor.—If A C is equal to B C, and therefore to B D, we have the square on A D equal to the square on A C, and eight times the square on D C; that is, to nine times the square on A C. Hence if a straight line be divided into three equal parts, the square on the whole line is equal to nine times the square on any one of the parts.

Prop. 9 thus:—Let the straight line A B be

divided into two equal parts at the point C, and into two unequal parts at the point D; then the squares

on A D, D B are together equal to double the squares on A C, C D.

The square on A D is equal to the squares on A C, C D together with twice the rectangle A C, C D; that is, the square on A D is *greater than* the squares on C B, C D by twice the rectangle C B, C D. And the square on D B is *less than* the squares on C B, C D by twice the rectangle C B, C D (Prop. 7, 2nd enunciation). Hence, adding,—the squares on A D and D B are together equal to double the squares on C B, C D.

Prop. 10 thus:—Let the straight line A B be bisected in C, and produced to D; then the squares

on A D, D B shall be together double of the squares on A C, C D.

Produce D A to E, making A E equal to B D; so that E C is equal to C D and E B to A D. Hence, by the preceding proposition, the squares on E B, B D are together double the squares on E C, C B. That is, the squares on A D, B D are together double the squares on C D, A C.

Props. 9 and 10 may be included under one enunciation thus:—

The squares on two lines (A D and D B) are together double the squares on half the sum and half the difference of the two lines ; or thus :—

The squares on the sum and difference of two lines (A C and C D) are together double the squares on the two lines.

COR.—Since the squares on A D, D B exceed the squares on A C, C B by twice the square on C D, it follows that when a straight line is bisected the sum of the squares on the two parts is least, and the sum is greater as the difference between the two parts of the divided line is greater.

It is well to notice the algebraical and arithmetical relations which the different properties presented in the preceding propositions serve to illustrate.

We must show first that if each of the two lines which contain a rectangle can be divided into an exact number of parts, each equal to some unit of linear measurement, then the product of the two numbers represents the number of corresponding units of square measurement contained in the rectangle.

Let the rectangle A B C D be contained by the lines A B, A D; and suppose that a certain unit of length is contained 13 times in A B and 7 times in A D. Then if A B be divided into 13 equal parts and A D into 7 equal parts, each part of each line is equal to this unit of length. And if we draw through the points of division in A B lines parallel to A D, and through the points of division in A D lines parallel to

A B, it is clear that the rectangle A B C D will be divided into a number of squares each having its sides equal to the unit of length. Now each row of squares parallel to A B contains 13 such squares, and there are seven such rows. Therefore the whole rectangle contains 7 times 13 squares. Thus the product of the numbers 7 and 13, which represent the length of the sides in terms of the linear unit, gives us the number representing the area of the rectangle in terms of the corresponding unit of square

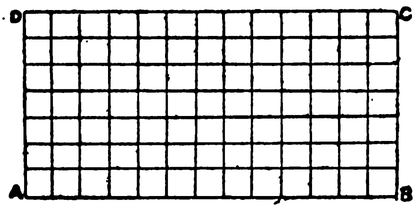


FIG. 81.

measurement. And the proof would have been precisely the same whatever the number of units of linear measurement in the sides A B, A D—so that if A B contains a such units, and A D contains b, the rectangle A B C D contains $a\ b$ units of square measurement.

It would be easy to extend this proof to the case of a rectangle having incommensurable sides; but for the purpose of illustration the case of commensurable sides is sufficient. This commensurability is to be understood as implied in what follows.

In Euc., Book II., Prop. 1, if the undivided line contain a units of length, the several parts of the divided line b, c, and d units, respectively, the proposition corresponds to the algebraical identity

$$a(b + c + d) = a\,b + a\,c + a\,d.$$

Prop. 2.—If the undivided line contain $(a + b)$ units of length, its parts a and b units, this proposition corresponds to the identity

$$(a + b)\,a + (a + b)\,b = (a + b)^2.$$

In Prop. 3, on the same supposition, the algebraical identity corresponding to the proposition is

$$(a + b)\,b = a\,b + b^2.$$

Prop. 4, on the same supposition, corresponds to the identity

$$(a + b)^2 = a^2 + 2\,a\,b + b^2.$$

In Prop. 5, let $AB = 2a$, and $CD = b$, so that $AD = (a + b)$ and $DB = (a - b)$, then the corresponding algebraical identity is

$$(a + b)\,(a - b) + b^2 = a^2;$$

that is, the well-known relation

$$a^2 - b^2 = (a + b)\,(a - b).$$

But if we put $AD = a$ and $DB = b$, we obtain the relation—

$$a\,b + \left(\frac{a - b}{2}\right)^2 = \left(\frac{a + b}{2}\right)^2;$$

that is, the well-known formula—

$$(a + b)^2 - (a - b)^2 = 4\,a\,b.$$

We get the same identities in the case of Prop. 6 if we make corresponding suppositions, simply interchanging a and b.

In Prop. 7, put A C $= a$, and B C $= b$, then the algebraical identity corresponding to the proposition is

$$(a + b)^2 + b^2 = a^2 + 2b\,(a + b).$$

In Prop. 8, put A C $= a$, and B C $= b$; then the corresponding algebraical relation is

$$4\,(a + b)\,b + a^2 = (a + 2b)^2.$$

In Prop. 9, put first A B $= 2\,a$ and C D $= b$; then the corresponding algebraical identity is

$$(a + b)^2 + (a - b)^2 = 2\,(a^2 + b^2).$$

Next put A D $= a$ and D B $= b$, and we obtain the relation

$$a^2 + b^2 = 2\left(\frac{a + b}{2}\right)^2 + 2\left(\frac{a - b}{2}\right)^2,$$

which is not a new relation, the change in our suppositions merely leading to the inversion of the former relation.

In Prop. 10, corresponding suppositions with the interchange of a and b give the same results.

Any theorem respecting rectangles may be shown to correspond to an algebraical identity; and in like manner any homogeneous algebraical identity of two dimensions may be made to supply one or more geometrical theorems respecting rectangles.

Let us take as an instance the following identity:—

$$(a + b + c)^2 = a^2 + b^2 + c^2 + 2\,ab + 2\,bc + 2\,ca.$$

This resolves itself into the following proposition:—

PROP. I.—*If a straight line A B be divided into any three parts in the points C and D, then the*

square on A B shall be equal to the squares on A C, C D, and D B together with twice the rectangles contained by A C, C D, by A C, D B, and by C D, D B.

By Euc. II., Prop. 4, the square on A B is equal to the squares on A C, C B, together with twice the

rectangle A C, C B; that is (again applying Prop. 4), to the squares on A C, C D, D B together with twice the rectangle C D, D B and (Prop. 1) twice the rectangles A C, C D, and A C, D B.

Prop. 11 is an important one. It may be enunciated also thus:—*To divide a given straight line into two parts so that the squares on the whole line and on one of the parts may be together equal to three times the square on the other part.* That this enunciation is equivalent to the other follows immediately from Prop. 7. Prop. 11 offers a problem somewhat more difficult than most of those in Euclid. It is made use of by him in Book IV., Prop. 10; but when it is required for the solving of Prop. 30, Book VI., he appears to have forgotten that he had already solved it, and, adopting a less happy mode of analysing it, occupies three long propositions with its solution. The following is an analogous proposition.

PROP. II.—*To produce a given straight line A B so that the rectangle contained by the whole line thus produced and the given straight line may be equal to the square on the part produced.*

Produce A B to C and D, making B C equal to C D equal to A B. Divide C D in E so that the rectangle C D, D E may be equal to the square on C E. Then the rectangle A E, A B shall be equal to the square on B E.

For, the square on B E is equal to the squares on B C, C E together with twice the rectangle B C, C E; that is, to the square on A B, the rectangle C D, D E (*const.*), and twice the rectangle A B, C E;

that is, to the rectangle contained by A B and the line made up of A B, D E, and twice C E. But the sum of these lines is equal to A B, C D, and C E together—that is, to A B, B C, and C E, or to A E. Hence the square on B E is equal to the rectangle A E, A B.

Props. 11 and II. correspond to the two solutions of the quadratic

$$a\ (a-x) = x^2,$$

which results as the analytical expression of the relation in Prop. 11, when A B is made equal to a, and the smaller section of A B equal to x.

Props. 12 and 13 are important in solving geometrical problems of a certain class, though Euclid himself makes no use of these propositions. Each has a *general* and also an *exact* converse theorem. The general theorem converse to Prop. 12 is this:—

If the square on one side of a triangle is greater than

the sum of the squares on the other two sides, these two sides contain an obtuse angle. The proof is simple: the angle contained by the two sides must either be acute, right, or obtuse. If it were acute, then by Euc. X., Book II., 13, the squares on the sides containing this angle would together be greater than the square on the remaining side; but they are not greater: if it were right, the squares on the sides containing this angle would together be equal to the square on the remaining side; but they are not equal to this square. Therefore the angle is obtuse.

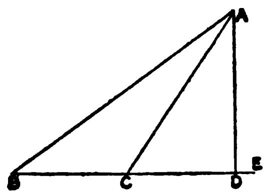


FIG. 82.

And in like manner may be proved the theorem converse to Prop. 13, viz.:—*If the square on one side of a triangle be greater than the sum of the squares on the remaining sides, these sides contain an acute angle.*

These two propositions may be referred to as Euc. II., Props. 12 and 13, *gen. conv.*

The exact converse theorem to Prop. 12 is this:—*If A C B be an obtuse angle, and B C be produced to D, so that the squares on B C and A C with twice the*

rectangle B C, C D are equal to the square on A B, then A D is perpendicular to B D. This property is often useful, as is the corresponding property converse to Prop. 13, viz.:—*If A B C be an acute angle and a point D is taken in B C (produced if necessary), such that the squares on A B and B C together exceed the square on A C by twice the rectangle B C, B D, then*

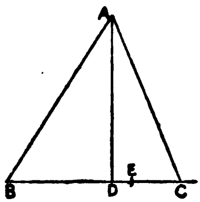


FIG. 83.

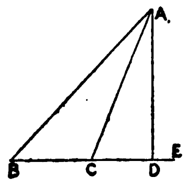


FIG. 84.

A D is perpendicular to B D. The proof in either case is easy, for, in the first case, if the foot of the perpendicular from A on B C produced fell otherwise than at D—at E suppose, it can be readily shown to follow from Prop. 12 that C D is equal to C E; which is absurd: and similarly in the second case we can show (if E is the foot of the perpendicular from A) that B E is equal to B D; which is absurd.

These propositions may be referred to as Euc., Book II., Props. 12, 13, *exact conv.*

SECTION III.

RIDERS AND PROBLEMS ON THE FIRST TWO BOOKS.

I. EASY RIDERS ON EUCLID'S FIRST THIRTY-FOUR PROPOSITIONS, WITH SUGGESTIONS FOR SOLUTION.

PROP. 1.

1. On a given straight line describe an isosceles triangle having each of the sides equal to a given straight line.

PROP. 2.

2. Show that there are in general eight different cases in the solution of this problem; and without drawing in the complete figure for each case show where the different lines will fall.

It will be found that if the given point be connected with either extremity of the given line, or if the equilateral triangle be described on either side of the line thus drawn, or if those sides of the equilateral triangle which pass through the given point be produced either way, a solution results.

3. If the diameter of the smaller circle is the radius of the larger, show that the given point and the vertex of the constructed triangle lie on the circumference of the smaller circle.

PROP. 3.

4. Having drawn two unequal lines, go through the complete construction involved in the method of Prop. 3; showing that in this construction five circles appear, and that there are two pairs of equal circles.

PROP. 4.

5. If two straight lines bisect each other at right angles, any point in either is equidistant from the extremities of the other.

6. Apply the method of superposition to establish the first case of Prop. 26.

7. The line which bisects the vertical angle of an isosceles triangle also bisects the base.

8. Let A B be a given straight line, and from A let equal straight lines A C, A D be drawn, making equal angles with A B on opposite sides of it; show that A B, produced if necessary, bisects C D at right angles.

9. The triangle A B C has equal sides A B and A C, and A D bisects the angle B A C, the point D not lying in B C. Show that if lines D B E and D C F are drawn so that B E is equal to C F, then the triangle A E F is isosceles.

10. The sides A B, A D of a quadrilateral are equal, and the diagonal A C bisects the angle D A B. Show that the sides B C, C D are equal, and that the diagonal A C bisects the angle B C D.

Prop. 5.

11. A B C is an isosceles triangle having each of the angles B and C double of the angle A. B D is drawn bisecting the angle B and meeting A C in D; show that B D is equal to A D.

12. Two straight lines A B and C D intersect in E and the lines E A, E B, E C, and E D are all equal. Show that the four angles E A D, E D A, E C B, and E B C are all equal.

Show that the triangles B A C, D C A *are equal in all respects by Euc.* I., 4.

13. Apply the preceding proposition to show that when two straight lines intersect the vertical angles are equal, without assuming any proposition beyond the fifth.

14. In the quadrilateral A B C D, A B is equal to A D and B C to C D; show that the angle A D C is equal to the angle A B C.

Prop. 6.

15. In the figure of I. 5, if F C, B G meet in H, the triangle B H C is isosceles.

16. If, further, F G is drawn, the triangle F H G is isosceles.

17. If, further, A H be drawn, the triangles A B H, A C H are equal in all respects.

18. With the same construction A H bisects B C at right angles.

19. With the same construction A H produced bisects F C at right angles.

20. With the same construction the triangles B H F, C H G are equal in all respects.

If Problems 15–20 be taken in order, the student will find no difficulty in solving them without using any propositions beyond Euc. I., 6.

21. If the angles A B C, A C B at the base of an isosceles triangle be bisected by the straight lines B D, C D, show that D B C will be an isosceles triangle.

22. In the quadrilateral A B C D, D C is equal to B C, and the angle A B C is equal to the angle A D C. Show that A D is equal to A B.

Join D B and apply Euc. I., 5 *; the rest is obvious.*

Prop. 8.

23. The diagonals of a rhombus intersect each other at right angles.

24. A quadrilateral has two of its opposite sides equal, and its diagonals are also equal. Show that the diagonals divide the quadrilateral into four triangles, whereof two are isosceles and the other two equal to each other in all respects.

25. From every point of a given line the lines

drawn to each of two given points on opposite sides of the line are equal. Prove that the line joining the given points will be bisected by the given line at right angles.

26. Show how Prop. 8 may be established without the use of Prop. 7, by applying the base of one triangle to the base of the other, the equal sides being conterminous but the vertices lying on opposite sides of the base.

Join the vertices; the rest is obvious.

PROP. 9.

27. If the base angles of an isosceles triangle be bisected, and the point of intersection of the bisectors joined to the vertex of the triangle, show that the vertical angle is bisected by the line thus drawn.

PROP. 10.

28. Show how to bisect a given straight line without making use of any proposition beyond the sixth.

See fourth rider to Prop. 6.

PROP. 11.

29. Show how to draw a straight line at right angles to a given straight line from a given point in the same without making use of any proposition beyond the sixth.

30. Find a point in a given line such that its distances from two given points may be equal.

31. Describe a circle of given radius to pass through two given points.

PROP. 12.

32. Two straight lines are drawn from a given point. From another given point it is required to draw a straight line which shall cut off equal parts from the given straight lines.

33. From two given points on opposite sides of a given straight line, draw two straight lines which shall meet in that given straight line and include an angle bisected by that given straight line.

34. From two given points on the same side of a given straight line, draw two lines which shall meet in that line and make equal angles with it.

35. A B, A C are two given straight lines, and D is a given point; it is required to find a point E in A B and a point F in A C, such that the lines D E, F E shall make equal angles with A B, and the lines D F, E F with A C.

Notice that D and F bear the same relation to A B that is investigated in 34*; and in like manner D and E bear the same relation to A C. These considerations will suggest a construction requiring the use of Prop.* 12.

36. A B, A C are two given straight lines, and D is a given point so placed that the perpendicular from D on A C cuts A B. It is required to find a point E in A B and a point F in A C, such that D F and D E

make equal angles with A C, and A E bisects the angle D E F.

Apply 33 *and* 34.

PROP. 13.

37. In the second figure to Prop. 13, if lines be drawn bisecting the angles A B C, A B D, these lines shall be at right angles to each other.

PROP. 14.

38. If there be three lines A B, A C, and A D meeting in a point, and if the lines bisecting the angles B A C, C A D are at right angles to each other, then shall A B and A D lie in the same straight line.

PROP. 15.

39. If four straight lines meet at a point so that the vertical angles are equal, these straight lines are two and two in the same straight line.

40. If A B C, D B E are two straight lines intersecting in B, and A B is equal to B D, B E to B C; show that the quadrilateral A D C E is made up of four triangles, whereof two are isosceles and the other two equal in all respects.

41. If with the same construction A B is equal to B C and D B to B E, the quadrilateral A D C E is made up of four triangles of which each opposite pair are equal in all respects.

PROP. 16.

42. In the triangle A B C, A D is drawn bisecting the angle B A C, and meeting B C in D; show that the angle B D A is greater than the angle B A D.

43. Through D, a point in the base B C of an isosceles triangle A B C, a line E D F is drawn meeting A B in E and A C produced in F; show that the angle A E F is greater than the angle A F E.

44. In the figure of Prop. 17, show that the angles A B C and A C B are less than two right angles, without producing B C.

Join A to a point in B C, and apply Prop. 16 *twice.*

PROP. 17.

45. If two sides of a triangle are produced, show that the two exterior angles thus formed are together greater than two right angles.

46. Show that any three angles of a quadrilateral are together less than four right angles.

PROP. 18.

47. Each of the diagonals of a quadrilateral figure exceeds the greatest side; show that the sum of two opposite angles exceeds half the sum of the remaining angles.

48. A B C D is a quadrilateral of which A D is the longest side and B C the shortest; show that

the angle A B C is greater than the angle A D C, and the angle B C D greater than the angle B A D.

Prop. 19.

49. In the figure of Euc. I., 5, show that F C is greater than B C.

50. Either diagonal of a rectangle exceeds the greatest side.

51. Either diagonal of a rectangle exceeds any line which has one extremity at an angle, and the other on a side of the rectangle.

52. The perpendicular is the shortest straight line that can be drawn from a given point to a given straight line; and of others that which is nearer to the perpendicular is less than the more remote; and only two equal straight lines can be drawn from the given point to the given straight line, one on each side of the perpendicular.

53. P and Q are points on the same side of the line A B; P C is drawn perpendicular to A B, and produced to D so that C D is equal to P C; show that Q D is greater than P Q.

Let Q D cut A B in E, and join E P: with this construction the proof is obvious.

Prop. 20.

54. The difference of any two sides of a triangle is less than the third side.

55. A point P is taken within the triangle

A B C; show that the sum of the distances P A, P B, and P C is greater than half the sum of the sides of the triangle.

56. With the same construction as in Ex. 53, a point F is taken in A B; show that the sum of the lines P F and Q F is greater than Q D.

57. A B C is a triangle having the angle B obtuse. A point D is taken in B C, and in A D, D E is taken equal to A B, and E A is bisected in F. Show that C F and D F are together greater than C A, A B.

58. The diagonals of a quadrilateral are together less than the sum of any four straight lines that can be drawn to the four angles of the quadrilateral from any point whatever except the intersection of the diagonals of the quadrilateral.

59. The sides of a quadrilateral are together greater than the two diagonals together.

60. In the triangle A B C the line B D is drawn bisecting the vertical angle A B C. If any point E is taken in B D, show that the difference of the sides A B, B C exceeds the difference of the lines A E, E C.

From A B the greater of the sides A B, B C (suppose) cut off B F equal to B C the less; show that E F is equal to E C, and apply Ex. 54.

61. With the same construction, show that if the point E lies in B D produced either way, the difference of A B, B C exceeds the difference of A E, E C.

62. A straight line A B is divided into two unequal parts in the point C, and a straight l ne C D is drawn at right angles to A B. Show that the difference of the lines A D, B D is less than the difference of the lines A C, B C.

Prop. 21.

63. A point P is taken within the triangle A B C; show that the sum of the lines P A, P B, and P C is less than the sum of the sides of the triangle.

64. A B C D is a quadrilateral whose diagonals intersect in E, and a point F is taken within the triangle A B E. Show that the sum of the diagonals A C, B D, together with twice the side A B, exceeds the sum of the four lines A F, B F, C F, and D F.

Prop. 23.

65. If one angle of a triangle be equal to the sum of the other two, the triangle can be divided into two isosceles triangles.

66. Construct a triangle having given the base and the two angles adjacent to the base.

67. Construct a triangle having given the base, an angle adjacent to the base, and the sum of the two sides.

68. Construct a triangle having given the base, an angle adjacent to the base, and the difference of the two sides—first when the greater side is adjacent

to the given angle, secondly when it is opposite to that angle.

PROP. 24.

69. Prove that the point F in the figure to this proposition falls below E G.

70. Show how the proof of the proposition may be completed without assuming that F falls below E G.

In dealing with the assumption that F may fall within the triangle D G E, apply Euc. I., 21.

PROP. 26.

71. A E B, C E D are two straight lines intersecting in C; A E is taken equal to E B, and lines A D, B C are drawn in such a way that the angles E A D, E B C are equal. Show that E C is equal to E D.

72. If from any point in a line bisecting a given angle perpendiculars be drawn on the lines containing the angle, these perpendiculars shall be equal, and shall cut off equal parts from those lines.

73. In a given straight line find a point such that the perpendiculars from it on two given straight lines shall be equal.

74. Through a given point draw a straight line so as to cut off equal parts from two straight lines which meet in a point.

75. If the straight line which bisects the vertical angle of a triangle is perpendicular to the base, the triangle is isosceles.

76. Through a given point draw a straight line such that the perpendiculars on it from two given points may be on opposite sides of it and equal to one another.

PROPS. 27 AND 28.

77. Two straight lines A E B, C E D bisect each other in E; show that A C is parallel to B D.

PROP. 29.

78. From the centres, A and B, of two circles parallel radii A P, B Q are drawn; P Q meets the circumferences again at R and S; show that A R is parallel to B S.

79. If a straight line be drawn parallel to one o the sides of an equilateral triangle, it will form with the other sides, produced if necessary, another equilateral triangle.

80. If a straight line be drawn parallel to one of the sides of a triangle, it will form with the other sides, produced if necessary, a triangle equiangular to the first.

81. Two straight lines A E B, C E D intersect in E; two other straight lines A F and C G are parallel respectively to C D and A B; show that the angle A is equal to the angle C.

82. The point P lies between two parallel lines. Show that if any straight line through P terminated by the parallels is bisected in P, every straight line so drawn will be bisected in P.

83. The intersecting straight lines A E B, C E D, terminated by parallel lines A C and B D, bisect each other in E; show that A C is equal to B D.

84. The line drawn through the vertex parallel to the base of an isosceles triangle is perpendicular to the line bisecting the vertical angle.

85. If the line bisecting the exterior angle of a triangle be parallel to the opposite side, the triangle is isosceles.

86. If from any point in the bisector of a given angle lines be drawn parallel to and terminated by the lines containing the given angle, the lines thus drawn shall be equal and shall cut off equal parts from the others.

87. A B C is a triangle right angled at B, and D is the middle point of A C. Show that if the line E B F is parallel to A C, then the angle E B A is equal to the angle A B D, and the angle D B C to the angle C B F.

PROP. 30.

88. The parallel lines A B, C D, and E F are intersected by the line B D F G; show that the bisectors of the angles A B G, C D G, and E F G are parallel to each other.

89. With the same construction as in Ex. 81, show that the lines bisecting the angles A, C, and A E C are parallel to each other.

Prop. 31.

90. From a given point without a given line draw a line which shall make a given angle with the given line.

91. Draw a line D E parallel to the base B C of the triangle A B C, so that D E shall be equal to B D.

92. A B C is an isosceles triangle. Determine points D, E in A B, A C respectively (these being the equal sides of the triangle) such that the lines B D, D E, and E C may be equal to each other.

93. Draw a line D E parallel to the base C B of a triangle A B C so that D E may exceed C E by a given length.

94. A B C is an isosceles triangle, and the points D and E lie in A B and A C produced; show that if B E is parallel to the bisector of the angle A C B, and C D parallel to the bisector of A B C, D B, B C, and C E are equal to each other.

95. Draw a line D E parallel to the base B C of the triangle A B C, so that B D and C E together shall be equal to the line D E.

Suppose the line drawn, and take a point P in it such that D P is equal to B D, and therefore E P to C E. Notice that the triangles B D P and C E P are isosceles, &c.

96. Draw a line D E parallel to the base B C of

the triangle A B C, so that D E shall be equal to the difference of B D and C E.

Suppose D E drawn as required, and produce D E to P, making D P equal to D B (supposed greater than C E). Then E P is equal to E C, and the triangles B D P and C E P are isosceles, &c.

Prop. 32.

97. If one angle of a triangle is equal to the sum of the other two, the triangle is right-angled.

98. If one angle of a triangle be greater than the sum of the other two, the triangle is obtuse-angled.

99. In an acute-angled triangle the sum of any two angles is greater than the third angle.

100. If the base of an isosceles triangle be produced, the exterior angle exceeds a right angle by half the vertical angle.

101. If the base of a triangle be produced either way, the sum of the two exterior angles thus formed exceeds two right angles by the vertical angle of the triangle.

102. If the three sides of a triangle be produced either way, as in the preceding example, the sum of the six exterior angles thus formed is equal to eight right angles.

103. If F G be joined in the figure to Euc. I., 5, B C and F G are parallel.

104. In the figure to Euc. I., 8, the difference

between the angles D and G is equal to the difference between the angles D E G and D F G.

105. In the figure to Euc. I., 21, the angle B D C exceeds the angle B A C by the sum of the angles A B D and A C D.

106. In the triangle A B C, B D and C D are drawn bisecting the angles A B C, A C B. Show that the angles B A C, D B C, and D C B are together equal to the angle B D C.

107. With the same construction, show that the angle B D C exceeds a right angle by half the angle B A C.

108. Determine the magnitude of the angles of a regular pentagon.

109. Show that the interior angle of a regular figure of n sides exceeds a right angle by $\frac{n-4}{n}$ths of a right angle.

110. On the sides of any triangle A B C equilateral triangles B C D, C A E, A B F are described, all external to A B C. Show that the lines A D, B E, C F are all equal.

Establish the equality of the triangles A C D, B C E, &c.

111. In the triangle A B C, the lines B D, C E are drawn perpendicular to A C, A B respectively. Show that the angle A B D is equal to angle A C E.

112. With the same construction, show that the angle E F D exceeds the angle E A D by twice the angle A B D.

113. Show also that the angle D F C is equal to the angle B A C.

114. Show also that the angles B F C, B A C are together equal to two right angles.

115. If the straight lines bisecting the angles at the base of an isosceles triangle be produced to meet, they will contain an angle equal to an exterior angle of the triangle.

116. Show that every right-angled triangle may be divided into two isosceles triangles.

117. If A B C be a straight line, bisected in B, and any line B D equal to A B or B C be drawn from B, show that A D C is a right-angled triangle.

118. Trisect a right angle.

119. Construct an isosceles triangle, having the vertical angle equal to four times the angle at the base.

120. One of the acute angles of a right-angled triangle is three times as great as the other; trisect the smaller of these.

121. On a given straight line A B an equilateral triangle A C B is described, the angles A and B are bisected by lines meeting in D, and lines D E, D F are drawn parallel to the lines A C and B C respectively. Show that the line A B is trisected in the points E and F.

122. Construct an isosceles triangle which shall have one-third of each angle at the base equal to half the vertical angle.

123. Construct a triangle having angles equal to

those of a given triangle, and the sum of the sides containing a given angle equal to a given straight line.

Suppose A B C to be the required triangle, so that A B and B C together may be equal to a given straight line. Produce A B to D, making B D equal to B C; then it will be found that enough is known about the triangle A D C to enable us to construct it, and hence to construct the required triangle.

124. The hypotenuse of a right-angled triangle is equal to twice the distance separating the right angle from the bisection of the hypotenuse.

125. Perpendiculars are let fall from two angles of a triangle upon the opposite sides. Show that their feet are equidistant from the bisection of the side opposite the remaining angle of the triangle.

126. A B C is an equilateral triangle, and B D is drawn perpendicular to the base A C; a point E is taken in B D so that E A, E B, and E C are all equal. Show that the angle A E C is equal to four times the angle E A D.

127. With the same construction, if D B is produced to F so that B F is equal to A B or B C, then the angle F A C is equal to two and a half times the angle A F C.

128. In the base B C of an isosceles triangle A B C a point D is taken, and a point E is taken so that C E is equal to D C. Show that three times the angle A E F is greater than four right angles by the angle A F E.

Begin by showing that three times the angle A E F is equal to four right angles increased by the angle E C D and diminished by the angle E D C. This follows readily from the fact that the angle A E F is equal to the two angles E C D and E D C together. The rest is obvious.

129. In the triangle A B C the side B C is bisected at E and A B at F; A E is produced to G so that E G is equal to A E, and C F is produced to H, so that F H is equal to C F; show that the points G, B, and H are in one straight line.

It may be shown that the angle H B A is equal to the angle B A C, &c.

130. Construct a right-angled triangle, having given the hypotenuse and the sum of the sides.

Suppose the triangle A B C to be constructed as required, A B being the hypotenuse; then, if A C be produced to D so that C D is equal to C B, the triangle B C D has two angles each equal to half a right angle. Therefore in the triangle A B D we have given us A B, A D, and the angle D. This suffices for the solution of the problem.

131. Construct a right-angled triangle, having given the hypotenuse, and the difference of the sides.

The analytical treatment resembles that of Ex. 130.

132. Construct a right-angled triangle, having given the hypotenuse and the perpendicular from the right angle on the hypotenuse.

Construct first a right-angled triangle A B C, C being the right angle, A B equal to half the hypotenuse

of the required triangle, and B C equal to the given perpendicular; produce A C (both ways) to D and E, so that A D and A E may each be equal to A B. Then show that D B E is the required triangle.

133. Construct a right-angled triangle, having given the perimeter and an angle.

From the extremities of a line A B equal to the given perimeter draw lines A C, B C inclined to A B at angles respectively equal to half a right angle and to half the given angle. Draw C D perpendicular to A B. Then C D is a side of the required triangle. The rest of the construction and the proof will readily suggest themselves.

134. Construct a triangle of given perimeter, having its angles equal to those of a given triangle.

The method used in Ex. 129 *must be applied, with variations which will at once suggest themselves.*

135. Construct a triangle, having given one side, an angle opposite to it, and the sum of the remaining sides.

The method is the same as that of Ex. 130.

136. Construct a triangle, having given one side, an angle opposite to it, and the difference of the remaining sides.

The method is that of Ex. 131.

137. If in the sides of a square, at equal distances from the four angles, four points be taken, one in each side, the figure formed by joining these will be also square.

138. If the alternate angles of any polygon be

produced to meet, the angles formed by these lines, together with eight right angles, are together equal to twice as many right angles as the figure has sides.

139. A P, B P, and C P are the internal bisectors of the angles of the triangle A B C. A P is produced to meet B C in D, and P M is drawn perpendicular to B C; show that the angle B P D is equal to the angle C P M.

140. Construct a right-angled triangle having equal sides, its right angle and one of the remaining angles upon two given parallel lines, and the third angle at a given point.

Draw a perpendicular from the given point to the nearest parallel, and from the foot of this perpendicular measure off along the parallel a distance equal to the distance separating the parallels. The point thus indicated is the right angle of the required triangle.

141. Construct a right-angled triangle having equal sides, its right angle at a given point, and its other angles upon two given parallel lines.

This problem may readily be shown to depend on the preceding one.

Prop. 33.

142. Two straight lines A B and A C are drawn from a point A; and two other straight lines D E and D F from a point D. A B is equal and parallel to D E, and A C is equal and parallel to D F. Show that B E is equal and parallel to C F.

143. If a quadrilateral have two of its sides

parallel, and the other two equal but not parallel, any two of its opposite angles are equal to two right angles.

144. Two equal but not parallel lines make equal angles on the same side of a third line which joins their extremities. Show that the straight line which joins their other extremities shall make equal angles with the two first lines and be parallel to the third.

145. In the figure to Euc. I., 5, G L drawn perpendicular as to B C produced, is produced to M, so that L M is equal to L G. Show that B L is equal and parallel to F C.

PROP. 34.

146. The diagonals of a parallelogram bisect each other.

147. If two straight lines bisect each other, the straight lines joining their extremities form a parallelogram.

148. No two straight lines drawn from the extremity of the base of a triangle to the opposite sides can possibly bisect each other.

149. If the opposite sides of a quadrilateral figure are equal, the figure is a parallelogram.

150. If the opposite angles of a quadrilateral figure are equal, the figure is a parallelogram.

151. The two straight lines A B, A C intersect in A, and P is a point within the angle B A C. It is required to draw a straight line B P C so that B P may be equal to P C.

Suppose B P equal to P C; join A P and produce to D so that P D may be equal to A B. Then A B D C is a parallelogram, &c.

152. With the same construction, Q is a point without the angle B A C. It is required to draw Q B C so that Q B may be equal to B C.

Take a point E in A B produced, so that B E may be equal to A B. Then Q E C A is a parallelogram (if Q B be assumed equal to B C).

153. From a given point in one of two intersecting lines it is required to draw a line terminated by the second, and such that the line drawn from the point of intersection of the given lines to the bisection of the required line may make given angle with one of the given lines.

154. From a given point P it is required to draw three straight lines, P A, P B, and P C, equal respectively to three given straight lines and having their extremities A, B, and C in one straight line, and A B equal to B C.

Suppose the lines drawn as required, P B lying between P A and P C; then if P B be produced to D so that B D is equal to P B, P A D C is a parallelogram, &c.

155. Draw a straight line through a given point such that the part of it intercepted between two given parallels may be of a given length.

156. Draw a straight line through a given point lying between two parallels, so that the line may be terminated by the parallels, and divided by the

given point into two parts having a given difference.

157. If the diameters of a quadrilateral figure bisect the angles, the figure is a rhombus.

158. If one diameter of a parallelogram bisect opposite angles, it is a rhombus.

159. If the diameter of a parallelogram intersect at right angles, it is a rhombus.

160. Straight lines bisecting adjacent angles of a parallelogram intersect at right angles.

161. Straight lines bisecting opposite angles of a parallelogram having unequal sides are parallel to each other.

162. If the diameters of a parallelogram are equal, it is a rectangle.

163. If the diameters of a quadrilateral figure bisect the angles and are equal, the figure is a square.

164. Find a point such that the perpendiculars let fall from it on two given straight lines may be equal to one another.

165. Between two given straight lines draw a straight line which shall be equal to one straight line and parallel to another.

166. On A B, a side of the parallelogram A B C D, a parallelogram A F E B is described, so that E B is in the same straight line with B D, and F B with B C. Show that E B is equal to B D.

167. If, in 166, A B bisects the angle F B D, F B is equal to B D.

168. On A B, D C, opposite sides of a parallelo-

gram, equilateral triangles A B E and C F D are described towards the same parts; show that F E A D and F E B C are parallelograms.

169. If in Ex. 168, C F D and A E B are described towards opposite parts, then D E B F and C E A F are parallelograms.

170. From A, C, opposite angles of the parallelogram A B C D, are drawn the four lines, A F, A E, C G, C H, perpendicular respectively to the sides A D, A B, C B, and C D, and on the side remote from the parallelogram; also A F is equal to C G, and A E to C H. Show that E G H F is a parallelogram.

171. Equilateral triangles are described on the four sides of a parallelogram. Show that the vertices of these triangles fall on the angles of a parallelogram—

(i) When all the triangles are towards the same parts as the parallelograms.

(ii) When all the triangles are towards opposite parts.

(iii) When two triangles on opposite sides are towards the same parts, and the other two triangles towards opposite parts.

172. On the sides A B, B C, and C D of a parallelogram A B C D three equilateral triangles A B E, B C F, and C D G are described, A B E and C D G towards the same parts as the parallelogram and B F C towards opposite parts. Show that E F and F G are respectively equal to two diagonals of the parallelogram.

Show that the triangle B F E is equal in all respects to the triangle A B C.

Show that the same holds good if B F C lies towards the same parts as the parallelogram and A B E, C D G towards opposite parts.

173. In the parallelogram A B C D, the angle A D B is equal to the angle A C B. Show that A B C D is rectangular.

174. In the parallelogram A B C D, the angle A D B is equal to one-third part of the angle A E B; also A C and B D intersect at an angle equal to one-third part of two right angles. Show that one of the diagonals is at right angles to opposite sides of the parallelogram.

175. If the angle between two adjacent sides of a parallelogram be increased, while their lengths remain unchanged, the diagonal through the point of intersection will be diminished.

176. If two opposite sides of a parallelogram be bisected, the lines drawn from the points of bisection to the opposite sides trisect the diagonal.

177. If A B, a side of the parallelogram A B C D, be divided into n equal parts, show that a line drawn from C to the division point nearest to B cuts off from the diagonal B D one $(n+1)$th part,—measured from B.

Take for n *any convenient number—say* 7. *Divide A B and C D into* 7 *equal parts, and join C with the division nearest to B, the division nearest to C with the next division from B, and so on. It will then be easy,*

in the manner of the preceding example, to show that any one of the 8 parts into which the diagonal is thus divided is equal to any other part—or, in other words, that the diagonal is divided into 8 equal parts.

178. In the straight line A B C, A B is equal to B C. Show that perpendiculars drawn from the points A, B, and C upon any straight line meet it in equidistant points.

(i) When the line passes between A and C.

(ii) When the line does not pass between A and C.

179. In case (ii) of Example 178, show that the perpendicular from A and C are together double of the perpendicular from B.

180. In case (i) of Example 178, show that the difference of the perpendiculars from A and C is double of the perpendicular from B.

181. If straight lines be drawn from the angles of any parallelogram perpendicular to a straight line which is outside the parallelogram, the sum of those from one pair of opposite angles is equal to the sum of those from the other pair of opposite angles.

182. Determine a point in the base of a triangle from which lines drawn parallel to the sides, to meet them, are equal.

183. If an hexagonal figure admits of division into three parallelograms, each pair of opposite sides are equal and parallel.

Show that in general such an hexagonal figure

admits of being divided into three parallelograms in two different ways.

184. If each pair of opposite sides of a hexagon are parallel, and one pair equal, the other pairs are also equal.

185. If each pair of opposite sides of a hexagon be equal and parallel, the three straight lines joining opposite angles will meet in a point.

186. If each pair of opposite sides of a rectilinear figure having an even number of sides be equal and parallel, all the lines joining opposite angles meet in a point.

187. Describe a rhombus within a given parallelogram, so that one of the angular points may occupy a given point on the perimeter of the parallelogram.

188. Describe a rectangle within a given parallelogram, so that one of the angular points may occupy a given point on the perimeter of the parallelogram.

In Examples 187 *and* 188 *it suffices that the angles of the constructed figures should lie on the sides or the sides produced of the parallelogram. Previous examples show the relations which hold when a parallelogram is a rhombus or rectangular, and these will be found sufficient for the solution of Examples* 187 *and* 188.

189. The three sides of a triangle are together less than the three lines drawn from the angles to the bisections of the opposite sides.

Complete a parallelogram having two sides of the triangle as adjacent sides. Then show that these sides

are together greater than the diagonal which passes through the bisection of the base, &c.

II. PROBLEMS ON PROPOSITIONS 35 TO THE END OF BOOK I.

190. On the sides A B, A C of a triangle describe parallelograms A B D E, A C F G, and produce D E, F G to meet in H: then the area of these parallelograms together is equal to the area of the parallelogram on B C, whose side is equal and parallel to A H.

Draw this parallelogram, and show that H A produced divides it into parts equal respectively to A D and A F.

191. From a given point in one of the equal sides of an isosceles triangle draw a line, meeting the other side produced, which shall make with these sides a triangle equal to the given triangle. Let A B, A C be the equal sides; F the given point in A C; and let C D perpendicular to B C meet B A produced in D; draw C E parallel to F D, cutting B D produced in E; then F E A is the required triangle.

192. If one angle of a triangle be a right angle, and another be two-thirds of a right angle, show that the equilateral triangle on the hypothenuse is equal in area to the sum of those on the sides.

193. Convert a trapezium into a triangle of equal area with one angle common.

194. Given a triangle A B C and a point D in

A B; construct another triangle A D E equal to the former, and having the common angle A.

195. Change a triangle into another equal one of given altitude.

196. If the sides of any quadrilateral be bisected and the points of bisection joined, the included figure is a parallelogram, and equal in area to half the original figure; show also that the lines joining the bisections of opposite sides bisect each other.

There is a pretty statical proof of the last property resulting from the determination of the centre of gravity of four equal particles at A, B, C, and D.

197. Through D, E, the bisections of the sides A B, A C of a triangle, draw D F, E F parallel to B E, A B; and show that the sides of the triangle D C F are equal to the three lines drawn from the angles to bisect the sides.

198. Bisect a triangle by a line drawn from a given point in one of its sides.

199. If from any point in the diagonal of a parallelogram lines be drawn to the angles, the parallelogram will be divided into two pairs of equal triangles.

200. Through E, the bisection of the diagonal B D of a quadrilateral A B C D, draw F E G parallel to A C; and show that A G will bisect the figure.

201. A B C is a given triangle; draw B D, C E perpendicular to B C and on the same side of it, each equal to twice the altitude of the triangle;

bisect A B, A C in F, G; and show that the triangle A B C is equal to the sum or difference of the triangles B D F, C E G, according as the angles at the base of A B C are both or only one acute.

202. If of the four triangles into which the diagonals divide a quadrilateral, two opposite ones are equal, the quadrilateral has two opposite sides parallel.

203. Upon stretching two chains A C, B D, across a field, A B C D, I find that A C, B D make equal angles with C D, and that A C makes with A D the same angle that B C does with B D: hence prove that A B is parallel to C D.

204. The two triangles formed by drawing lines from any point between two opposite sides of a parallelogram to the extremities of those sides are together half the parallelogram.

205. The difference between two triangles formed by drawing lines from a point *not* between two opposite sides of a parallelogram to the extremities of those sides is equal to half the parallelogram.

206. If from the ends of one of the non-parallel sides of a trapezium two lines be drawn to the bisection of the opposite side, the triangle thus formed with the first side is half the trapezium.

207. In the figure, Euc. I., 47, show that if B G and C H be joined, these lines will be parallel.

208. In ditto, if D B, E C be produced to meet F G and K H in M, N, the triangles B F M, C K N are equiangular and equal to the triangle A B C.

209. In ditto, if G H, K E, F D be joined, each of the triangles so formed is equal to the given triangle A B C.

210. In ditto, produce F G, K H to meet in M, join M B, M C, and produce M A to cut B C in N; then show that M N is perpendicular to B C, and thence that the three lines A N, B K, C P intersect in one point.

III. PROBLEMS ON BOOK II.[1]

211. If a line be drawn from one of the acute angles of a right-angled triangle to the bisection of the opposite side, the square upon that line is less than the square upon the hypotenuse by three times the square upon half the line bisected.

212. If from the middle point of one of the sides of a right-angled triangle a perpendicular be drawn to the hypotenuse, the difference of the squares of the segments so formed is equal to the square of the other side.

213. In any triangle, if a perpendicular be drawn from the vertex to the base, the difference of the squares upon the sides is equal to the difference of the squares upon the segments of the base.

[1] These problems, as well as several in the last section of problems in Book I., are taken from the collection in Colenso's Euclid. But I have gone through all the fifty, and have made some necessary corrections. To the student who has gone carefully through the preceding pages none of these fifty problems will present any difficulty.

214. Let A O B be a quadrant of a circle, whose centre is O; from any point C in its arc draw C D perpendicular to O A or O B, meeting in E the radius which bisects the angle A O B: then show that the squares upon C D, E are together equal to the square upon O A.

215. If from any point in the diameter of a semicircle two lines be drawn to the circumference, one to the bisection of the arc, and the other perpendicular to the diameter, then the squares upon these two lines are together double of the square upon the radius.

216. If A be the vertex of an isosceles triangle A B C, and C D be drawn perpendicular to A B, prove that the squares upon the three sides are together equal to the square on B D, and twice the square on A D, and thrice the square on C D.

217. If from any point perpendiculars be dropped on all the sides of any rectilineal figure, the sum of the squares upon the alternate segments of the sides will be equal.

218. If from one of the acute angles of a right-angled triangle a line be drawn to the opposite side, the squares of that side and the line so drawn are together equal to the squares of the hypotenuse and the segment adjacent to the right angle.

219. Describe a square equal to the difference of two given squares.

220. Divide, when possible, a given line into two parts, so that the sum of their squares may be equal to a given square.

221. From D the middle point of A C, one of the sides of an equilateral triangle A B C, draw D E perpendicular on B C; and show the square upon B D is three-fourths of the square upon B C, and the line B E three-fourths of B C.

222. If from the vertex A, of a right-angled triangle B A C, A D be dropped perpendicular on the base, show that the rectangles contained by B C and B D, B C and C D, B D and C D are respectively equal to the squares upon A B, A C, A D.

223. Produce a given line so that the rectangle of the whole line produced and the original line shall be equal to a given square.

224. If on the radius of a circle a semicircle be described, and a perpendicular to the common diameter be drawn, the square of the chord of the greater circle, between the extremity of the diameter and the point of section of the perpendicular, will be double of the square of the corresponding chord of the lesser circle.

225. Divide a line in two points equally distant from its extremities, so that the square on the middle part shall be equal to the sum of the squares on the extremes; and show also that in this case the square of the whole line will be equal to the squares of the extreme parts together with twice the rectangle of the whole and the middle part.

226. Divide a line into two parts, so that the squares of the whole line and one of the parts shall be together double of the square of the other part:

and show that, by the same division, the square of the greater part will be equal to twice the rectangle of the whole and the lesser part.

227. Divide a straight line into two parts so that the sum of their squares may be the least possible.

228. Show that the sum of the squares upon two lines is never less than twice their rectangle, and that the difference of their squares is equal to the rectangle of their sum and difference.

229. Show that of the two algebraical expressions, $(a+x)(a-x) + x^2 = a^2$, $(a+x)^2 + (a-x)^2 = 2a^2 + 2x^2$, the first is equivalent to Props. 5 and 6, and the second to Props. 9 and 10, of Euc. II.

230. A B C D is a rectangle, E any point in B C, F in C D: show that the rectangle A B C D is equal to twice the triangle A E F together with the rectangle B E, D F.

231. If a line be divided into two equal and also into two unequal parts, the squares of the two unequal parts are together equal to twice the rectangle contained by these parts together with four times the square of the line between the points of section.

232. If from one of the equal angles of an isosceles triangle a perpendicular be dropped on the opposite side, the rectangle of that side and the segment of it between the perpendicular and base is equal to half the square upon the base.

233. A, B, C, D, are four points in the same line; E a point in that line equally distant from the middle of the segments A B, C D; F any other point

in A D: show that the squares of A F, B F, C F, D F, are together greater than the squares of A E, B E, C E, D E by four times the square of E F.

234. If from the extremities of any chord in a circle lines be drawn to any point in the diameter to which it is parallel, the sum of their squares is equal to the sum of the squares upon the segments of the diameter.

235. If the sides of a triangle be as 2, 4, 5, show whether it will be acute or obtuse angled.

236. In any isosceles triangle A B C, if A D be drawn from the vertex to any point in the base, show that the difference of the squares on A B and A D is equal to the rectangle B D and C D.

237. If in the figure, Euc. I., 47, the angular points be joined, the sum of the squares of the six sides of the figure so formed is equal to eight times the square of the hypotenuse.

238. If one angle of a triangle be four-thirds of a right angle, the square of the side subtending that angle is equal to the sum of the squares of the sides containing it together with the rectangle contained by these sides.

239. If A B C be a triangle, with the angles at B, C, each double of the angle at A, then the square of A B is equal to the square of B C together with the rectangle A B and B C.

240. In any triangle A B C, if B P, C Q be drawn perpendicular to A C, A B, produced if necessary, then shall the square of B C be equal to the rect-

angle A B, B Q together with the rectangle of A C, C P, or to the difference of these rectangles where only one of these straight lines A C, A B is produced.

241. In [22] show that the square of the perpendicular is equal to the square of the line between the perpendicular and the other equal angle, together with twice the rectangle contained by the segments of the side, if the vertical angle is acute, or to the same square diminished by twice the rectangle contained by these segments if the vertical angle is obtuse.

242. If from the right angle of a right-angled triangle lines be drawn to the opposite angles of the square described on the hypotenuse, the difference of the squares on these lines is equal to the difference of the squares on the two sides of the triangle.

243. In any triangle the squares of the two sides are together double of the squares of half the base, and of the line joining its middle point with the opposite angle.

244. If B D be drawn bisecting A C, one of the sides of the triangle A B C, in D, and A E be drawn perpendicular to the base, show that the square upon B D is equal to the sum or difference of the square upon the half of A C and the rectangle B C, B E, according as E lies in B C or in B C produced.

245. Any rectangle is half the rectangle contained by the diameters of the squares upon its two sides.

246. If from any point within a rectangle lines be

drawn to the angular points, the sums of the squares upon those drawn to the opposite angles will be equal.

247. The squares of the diagonals of a parallelogram are together equal to the squares of the four sides.

248. The squares of the diagonals of a quadrilateral are together less than the squares of the four sides by four times the square of the line joining the bisections of the diagonals.

249. The squares of the diagonals of any quadrilateral are together double of the squares of the two lines joining the bisections of the opposite sides.

250. The squares of the sides of any triangle are together triple of the squares of the distances of the angles from the point of intersection of lines drawn from them to the bisections of the opposite sides.

251. If two opposite sides of any quadrilateral be bisected, the sum of the squares of the other two sides together with the squares of the diagonals is equal to the sum of the squares of the sides bisected together with four times the square of the line joining the points of section.

252. If D E be drawn parallel to the base B C of an isosceles triangle A B C, then the square of B E is equal to the rectangle of B C, D E together with the square of C E.

253. The squares of the diagonals of a trapezium are together equal to the squares of its two non-parallel sides, with twice the rectangle contained by its parallel sides.

254. If B D, C E be squares described upon the sides A B, A C of any triangle, show that the squares of B C and D E are together double of the squares of A B and A C.

255. If squares be described on the three sides of any triangle, and the angular points of the squares be joined, the sum of the squares of the sides of the hexagonal figure thus formed will be equal to four times the sum of the squares of the sides of the triangle.

256. If two points be taken in the diameter of a circle equally distant from the centre, the sum of the squares of two lines drawn from these points to any point in the circumference will be constant.

257. The hypotenuse A B of a right-angled triangle A B C is trisected in the points D, E: show that, if C D, C E be joined, the sum of the squares on the sides of the triangle C D E is equal to two-thirds of the square on A B.

258. Divide a given line into two parts, so that their rectangle may be equal to a given square.

259. If the areas of a triangle and of a square be equal, the perimeter of the triangle will be the greater.

260. A B C D is a quadrilateral, E the middle point of the line joining the bisections of the diagonals; if with E as centre any circle be described, show that for every point P in this circle, $PA^2 + PB^2 + PC^2 + PD^2$ is constant, and equals $EA^2 + EB^2 + EC^2 + ED^2 + 4EP^2$.

PRINTED BY
SPOTTISWOODE AND CO., NEW-STREET SQUARE
LONDON

THE

'KNOWLEDGE' LIBRARY.

EDITED BY RICHARD A. PROCTOR.

HOW to PLAY WHIST: with the LAWS and ETIQUETTE of WHIST: Whist Whittlings, and Forty fully-annotated Games. By 'FIVE OF CLUBS' (R. A. Proctor). Crown 8vo. 5*s*.

HOME WHIST: an Easy Guide to Correct Play, according to the latest Developments. By 'FIVE OF CLUBS' (Richard A. Proctor). 16mo. 1*s*.

The POETRY of ASTRONOMY. A Series of Familiar Essays on the Heavenly Bodies. By RICHARD A. PROCTOR. Crown 8vo. 6*s*.

NATURE STUDIES. Reprinted from *Knowledge*. By GRANT ALLEN, ANDREW WILSON, THOMAS FOSTER, EDWARD CLODD, and RICHARD A. PROCTOR. Crown 8vo. 6*s*.

LEISURE READINGS. Reprinted from *Knowledge*. By EDWARD CLODD, ANDREW WILSON, THOMAS FOSTER, A. C. RUNYARD, and RICHARD A. PROCTOR. Crown 8vo. 6*s*.

The STARS in their SEASONS. An Easy Guide to a Knowledge of the Star Groups, in Twelve Large Maps. By RICHARD A. PROCTOR. Imperial 8vo. 5*s*.

The STAR PRIMER. Showing the Starry Sky Week by Week in Twenty-four Hourly Maps. Crown 4to. 2*s*. 6*d*.

The SEASONS PICTURED, in Forty-eight Sun-Views of the Earth, and Twenty-four Zodiacal Maps and other Drawings. By RICHARD A. PROCTOR. Demy 4to. 5*s*.

STRENGTH and HAPPINESS. By RICHARD A. PROCTOR. With 9 Illustrations. Crown 8vo. 5*s*.

ROUGH WAYS MADE SMOOTH: a Series of Familiar Essays on Scientific Subjects. By RICHARD A. PROCTOR. Crown 8vo. 6*s*.

OUR PLACE AMONG INFINITIES: a Series of Essays contrasting our Little Abode in Space and Time with the Infinities Around us. By RICHARD A. PROCTOR. Crown 8vo. 5*s*.

The EXPANSE of HEAVEN: a Series of Essays on the Wonders of the Firmament. By RICHARD A. PROCTOR. Crown 8vo. 5*s*.

PLEASANT WAYS in SCIENCE. By RICHARD A. PROCTOR. Crown 8vo. 6*s*.

MYTHS and MARVELS of ASTRONOMY. By RICHARD A. PROCTOR. Crown 8vo. 6*s*.

London: LONGMANS, GREEN, & CO.